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Abstract

We construct a multi-country affine term structure model that contains unspanned macroeconomic and foreign exchange risks. The canonical version of the model is derived and is shown to be easy to estimate. We show that it is important to impose restrictions (including global asset pricing, carry trade fundamentals and maximal Sharpe ratios) on the prices of risk to obtain plausible decompositions of forward curves. The forecasts of interest rates and exchange rates from the restricted model match those from international survey data. Unspanned macroeconomic variables are important drivers of international term and foreign exchange risk premia as well as expected exchange rate changes.

JEL classification: E43, F31, G12, G15
Bank classification: Asset pricing; Interest rates; Exchange rates

Résumé

Les auteurs construisent un modèle affine multinational fondé sur la structure par terme des taux d'intérêt. Ce modèle intègre les risques macroéconomiques et de change qui ne peuvent être quantifiés en observant la courbe de rendement. La version canonique du modèle est établie par dérivation et est facile à estimer. Les auteurs montrent qu'il faut imposer des restrictions (évaluation internationale des actifs, contexte fondamental d'exécution des stratégies de portage et ratios de Sharpe maximaux) aux prix du risque pour obtenir des décompositions plausibles des courbes des taux à terme. Les prévisions que le modèle produit pour les taux d'intérêt et les taux de change concordent avec celles tirées des résultats d'enquêtes internationales. Les variables macroéconomiques dont la valeur n'est pas liée à la courbe de rendement observée influent de façon importante sur les primes de terme et de risque de change à l'échelle internationale ainsi que sur les anticipations en matière de taux de change.

Classification JEL : E43, F31, G12, G15
Classification de la Banque : Évaluation des actifs; Taux d'intérêt; Taux de change

1 Introduction

The links between U.S. interest rates and macroeconomic fundamentals have been explored in a growing literature using affine models of the term structure.[1] An important development in this literature is the role of unspanned macroeconomic variables. A variable is unspanned if its value is not related to the contemporaneous cross section of interest rates but it does help forecast both future excess returns on the bonds (i.e., term structure risk premia) and future interest rates. Term structure models are used to identify the (offsetting) effects of the unspanned variables in the two components. The identification of the unspanned risks is important as the traditional spanned factors (e.g., level, slope and curvature) that are able to capture the cross section of interest rates are not able to completely explain the physical dynamics of the data. Consequently, there has been an extensive search conducted to find unspanned variables embedded in the U.S. term structure with Cochrane and Piazzesi (2005, 2008), Kim (2007), Cooper and Priestly (2009), Ludvigson and Ng (2009), Joslin, Priebsch and Singleton (2010) (JPS), Orphanides and Wei (2010), Barillas (2011), Chernov and Mueller (2011) and Duffee (2011) offering various candidates.[2]

In this paper, we show the important role of unspanned risks in explaining the links between global macroeconomic fundamentals and the cross section of international interest rates and exchange rates. We construct and estimate a multi-country, dynamic affine term structure model of the international bond and foreign exchange markets. The model incorporates real growth and inflation from all of the countries examined as unspanned macroeconomic variables. In addition, a large part of the variation in exchange rates is orthogonal to both bond yields and the macroeconomic variables. The additional assumption of unspanned exchange rate risk permits the model to match the higher levels of volatility found in the currency market (e.g., Brandt and Santa-Clara (2002), Anderson, Hammond and Ramezani (2010)).

We use our model to decompose the cross section of global yields into expectations of future short-term rates and international term structure risk premia. A similar decomposition can be applied to exchange rates. In order to obtain plausible decompositions, we show that it is important to impose a number of economic restrictions on the term structure and foreign exchange risk premia. New to the term structure and foreign exchange literature, we find that the restricted model's forecasts of interest rates and exchange rates match those from survey data. These results are surprising given prior work using surveys to construct forecasts based on "subjective" beliefs that may differ from model based ones (e.g., Frankel and Froot (1989), Froot (1989), Chinn and Frankel (2000), Gourinchas and Tornell (2004), Bacchetta, Mertens and van Wincoop (2009) and Piazzesi and Schneider (2011)).

[1]See Kozicki and Tinsley (2001), Ang and Piazzesi (2003), Diebold, Piazzesi and Rudebusch (2005), Kim and Wright (2005), Ang, Dong and Piazzesi (2007), Gallmeyer, Hollifield, Palomino and Zin (2007), Ang, Bekaert and Wei (2008), Rudebusch and Wu (2008), Bekaert, Cho and Moreno (2010), Bikbov and Chernov (2010), Rudebusch (2010), Piazzesi (2010), Gurkaynak and Wright (2010), Ang, Boivin, Dong and Loo-Kung (2011) and Duffee (2012) the citations therein.

[2]Researchers have also uncovered unspanned factors in bond market volatility (e.g., Collin-Dufresne, Goldstein and Jones (2009), Anderson and Benzoni (2010)). This paper focuses on conditional first moments.

Our restricted model with unspanned risks yields a number of novel insights. New to the term structure literature, our decomposition shows that it is the global component of the (unspanned) macroeconomic variables that drives term structure risk premia. Unspanned real growth and inflation account for over 50 per cent of the variation in short-run forward term premia in all of the countries examined. The macroeconomic variables also have a relatively large effect on foreign exchange risk premia. New to the foreign exchange literature, a large portion of the effect comes from the unspanned component of the variables. For example, at the one-year horizon, the unspanned component accounts for approximately 50 per cent of the variation in the U.S. dollar/Euro exchange rate risk premium. In addition, the unspanned components of the macroeconomic variables also explain a large portion of the movements in expected exchange rates, especially at short horizons. We view our results as suggestive for further research on the links between macroeconomic variables and exchange rates using modern asset pricing methods.[3]

While there are a number of papers that estimate two-country affine term structure models with exchange rate risks (i.e., Saa-Requejo (1993), Frachot (1996), Backus, Foresi and Telmer (2001), Dewachter and Maes (2001), Ahn (2004), Inci and Lu (2004), Brennan and Xia (2006), Dong (2006), Graveline (2006), Chabi-Yo and Yang (2007), Diez de los Rios (2009), Anderson, Hammond and Ramezani (2010), Egorov, Li and Ng (2011) and Pericoli and Taboga (2012)), there are very few attempts to incorporate spanned or unspanned macroeconomic variables in multi-country models due to the computational complexity of estimating international term structure models that preclude arbitrage.[4] To overcome this problem, we derive the canonical version of a Gaussian, no-arbitrage model by adapting the methodologies of Joslin, Singleton and Zhu (2011) (JSZ) and JPS to incorporate the cross section of international yield curves and exchange rates. The model uses principal components from the international cross section of yields as bond market state variables. We conduct a number of analyses to show that, in the sample of yield curves from the four countries that we examine, two of the components are global (i.e., a global level and a global slope factor) while the remaining six factors are local.[5] Our international canonical model embeds a number of new identifying restrictions on the risk-neutral dynamics which makes it easy to estimate. The resulting model fits the cross section of bond yields with root mean squared pricing errors of less than 10 basis points.

An important contribution of the paper is to show how imposing economic restrictions on the term structure and foreign exchange risk premia aids in identifying the contribution of the unspanned factors. We impose three sets of economic restrictions which come from theory external to the model. The first is "global asset pricing": in the cross section of bond returns, only the global level and global slope factors command risk premia. The

[3]We discuss the model's findings in light of the "exchange rate disconnect" literature below.

[4]There is another strand of the literature that uses time-series regressions to link yield curve variables to exchange rates changes over short and long horizons (e.g., Campbell and Clarida (1987), Bekaert and Hodrick (2001), Bauer (2001), Clarida, Sarno, Taylor and Valente (2003), Chinn and Meredith (2005), Boudoukh, Richardson and Whitelaw (2006), Bekaert, Wei and Xing (2007), and Ang and Chen (2010)). Still other approaches are possible (e.g. the quadratic model of Leippold and Wu (2007), the multi-country Nelson-Siegel factor model of Diebold, Li and Yue (2008)).

[5]Following Perignon, Smith and Villa (2007), we introduce a new method of conducting an inter-battery factor analysis using the EM algorithm that confirms the interpretation of the components as global factors.

second restriction is to assume that: (i) the bond market factors affect foreign exchange risk premia through the difference between the U.S. and foreign short-term interest rate; and, (ii) the macroeconomic variables enter in relative form. We label these combined conditions as "carry trade fundamentals". The third restriction is to reduce the prices of risk to obtain plausible implied Sharpe ratios for investments in the global bond and foreign exchange markets.

The combined assumptions of no-arbitrage pricing and unspanned risks (for the risk-neutral dynamics), along with the economic restrictions of global asset pricing, carry-trade fundamentals and maximal Sharpe ratios (for the prices of risks) yield long-run projections for international bond yields under the physical measure that are very different from their unrestricted counterparts.[6] As in the domestic model of Cochrane and Piazzesi (2008), the restrictions on the prices of risk allow the cross section of interest rates (i.e., the risk-neutral distribution) to provide a lot of information about the time-series dynamics of yields (i.e., the physical distribution). These restrictions are important: long-run projections of short-term interest rates from an unrestricted model are essentially flat. This would indicate that investors were anticipating much of the drop in interest rates that occurred in our sample. However, with our (restricted) model, long-run expectations of short-term rates become more volatile and investors anticipate a smaller portion (if any) of the decline in interest rates.

We provide a more formal evaluation of the restricted model's forecasts of interest rates and exchange rates by comparing them to the forecasts from surveys collected by Consensus Economics Inc. We find that forecasts from the restricted model's physical distribution are consistent with those from the survey data. Information in the bond market factors and the macroeconomic variables that is not contained in the model's forecasts has little additional explanatory power in matching the survey data. Our results thus suggest that financial market participants understand: (1) the interaction between term structure and foreign exchange risk premia; (2) the fact that the macroeconomic variables that drive risk premia are not spanned by the current cross section of interest rates; and (3) economic relationships such as global asset pricing, carry-trade fundamentals and maximal Sharpe ratios are part of asset price dynamics. Once these conditions are imposed, the survey data appear to be closely aligned with beliefs arising in rational, integrated global markets.

Our results are complementary to those found in the small literature on multi-country, no-arbitrage term structure models. Hodrick and Vassalou (2002) is, to the best of our knowledge, the first paper to consider more than two countries at the same time. They focus on a multi-country version of the Cox, Ingersoll and Ross (1985) class of term structure models to model the short-end of the yield curve for the U.S., Germany, Japan, and the U.K.. More recently, Sarno, Schneider and Wagner (2011) estimate a multi-country affine term structure model with latent factors. While they do not include macroeconomic variables in their model, they show that the estimated risk premia are correlated with them. We extend their analysis by showing how to impose restrictions on the prices of risk in order to match both bond and foreign exchange dynamics.

[6] The importance of using long-run projections as a way of distinguishing among models with similar short-run dynamics has been noted in Kozicki and Tinsley (2001) and Cochrane and Piazzesi (2008).

Graveline and Joslin (2010), building on the work of JSZ, present a no-arbitrage term structure model to analyze the joint dynamics of exchange rates and swap rates for the G-10 currencies. In their model, bond yields are affine functions of the principal component of yields in the same country, while we use global principal components. This assumption allows us to restrict the prices of risk using global asset pricing and assess how unspanned macroeconomic variables affects both bond and foreign exchange rate risk premia. Jotikasthira, Le and Lundblad (2010), also building on the work of JSZ and JPS, present a three-country term structure model to study how macroeconomic shocks affect current and expected short-term rates. While their model includes unspanned macroeconomic risks, our modeling framework and goals complements their approach.[7] First, we incorporate exchange rates in our estimation which allows us to analyze the implications of unspanned macroeconomic variables for exchange rate risk premia. Second, we analyze the impact of economic restrictions on the model. Finally, we compare the model's forecasts to those from survey data.

This paper is organized as follows. The next section introduces the notation and a preliminary analysis of the data. The asset pricing model is presented in section 3 while its estimation is discussed in detail in section 4. Section 5 contains the model's empirical results. Section 6 presents the model's forecasts of interest rates and exchange rates and compares them to the survey data. The important role of unspanned macroeconomic variables is also examined. The final section concludes. A separate appendix provides a number of technical details.

2 Preliminary analysis

2.1 Notation

We adapt the notation used in Cochrane and Piazzesi (2005) and Ludvigson and Ng (2009) to a multi-country analysis. Our analysis concerns a world with $J+1$ countries and currencies where, without loss of generality, we consider the $J+1$st currency to be the numeraire (U.S. dollar in our case). We assume that for each country j there is a set of n-period (default-free) discount bonds with prices in the local currency given by $P_{j,t}^{(n)}$ for $n = 1, ..., N$. The log yield on a bond is given by

$$y_{j,t}^{(n)} = -\frac{1}{n} \log P_{j,t}^{(n)}.$$

We also refer to country j's short-term interest rate, or short rate, as the yield on the bond with the shortest maturity under consideration, $r_{j,t} = y_{j,t}^{(1)}$. As such, $r_{\$,t}$ is the short-term, risk-free interest rate for a U.S. investor. As in Cochrane and Piazzesi (2005, 2008), both n and t will be measured in years while the data will be sampled at a monthly frequency.

The one-year excess return on a bond of maturity n is the gain from buying an n-year bond from country j and selling it one year later, financing the position at the short

[7] Other international papers with unspanned risks include Dahlquist and Hasseltoft (2011) who construct local and global versions of the Cochrane-Piazzesi predictive factor and Wright (2011) who examines unspanned inflation. However, both of these papers estimate individual country models across a number of countries.

rate. For example, the U.S. dollar excess return for holding an n-year zero-coupon bond denominated in U.S. dollars is defined as:

$$rx_{\$,t+1}^{(n)} \equiv \log\left(\frac{P_{\$,t+1}^{(n-1)}}{P_{\$,t}^{(n)}}\right) - r_{\$,t} = ny_{\$,t}^{(n)} - (n-1)y_{\$,t+1}^{(n-1)} - r_{\$,t}. \tag{1}$$

Similarly, we can compute the U.S. dollar excess return to holding an n-year zero-coupon bond denominated in currency j while hedging the foreign exchange rate risk as:

$$rx_{j,t+1}^{(n)} \equiv \log\left(\frac{P_{j,t+1}^{(n-1)}}{P_{j,t}^{(n)}}\right) - r_{j,t} = ny_{j,t}^{(n)} - (n-1)y_{j,t+1}^{(n-1)} - r_{j,t}. \tag{2}$$

Note that this return is equivalent to the local-currency excess return that a local investor would obtain from buying an n-year local (country j) bond and selling it one year later. We will interpret the expected value of the excess holding period returns on a bond as the bond's risk premium.

It may not be optimal for the international bond investor to fully hedge the foreign currency exposure of the position. If a partial hedge is undertaken, the investor will be exposed to foreign exchange risk. The excess return earned by a domestic investor for holding a one-year zero-coupon bond from country j is:

$$sx_{j,t+1} \equiv \log\left(\frac{S_{j,t+1}}{S_{j,t}}\right) + y_{j,t}^{(1)} - y_{\$,t}^{(1)} = \Delta s_{j,t+1} + r_{j,t} - r_{\$,t}, \tag{3}$$

where $s_{j,t}$ is the (log) spot exchange rate for country j in terms of the numeraire currency (U.S. dollar price of a unit of foreign exchange). The expected value of $sx_{j,t+1}$ is the foreign exchange risk premium. In our analysis below, we show that both bond and foreign exchange risk premia are determined in global markets.

2.2 Data and summary statistics

Our data set consists of monthly observations over the period January 1975 to December 2009 of the U.S. dollar bilateral exchange rates against the Canadian dollar, the German Mark/Euro, and the British pound, along with the appropriate continuously compounded yields of maturities one to ten years for these countries. We also include data on the annual headline CPI inflation rates and the annual growth rates of industrial production for each of the countries.[8] The exchange rates and macroeconomic data are from Datastream, while the global yield curve variables are from the BIS.

Summary statistics for one, two, five and ten-year yields, as well as the corresponding annual rates of depreciation of the exchange rates, inflation and growth are presented in Table 1. All variables are measured in per cent per year. Our statistics are consistent with those found in previous studies (e.g., Backus et al. (2001) and Bekaert and Hodrick (2001)). For example, while the rates of currency depreciation have lower means (in absolute value) than those on bonds, the former are more volatile than the latter. Bond

[8] Following Engel and West (2006), we replace the June 1984 outlier in the German industrial production index by the average of the May and July 1984 figures.

yields display a high level of autocorrelation, while the rates of depreciation do not. The rate of depreciation of the U.S. dollar against the Canadian dollar is less volatile than the rates of depreciation of the U.S. dollar against the other two currencies. The United Kingdom ranks first in terms of the highest (average) level of interest rates during the sample period, followed by Canada, the United States, and Germany. On average, yield curves tend to slope upwards, with long term yields being less volatile than short ones.

We focus on inflation and growth as our macroeconomic factors as they have been used in a large number of previous macro-finance term structure models (see cites above). We construct proxies for global inflation and growth by using a GDP-weighted average of the domestic inflation and growth rates.[9] Summary statistics for the individual country data are shown in Table 1.

2.3 Global bond market factors

It is well documented that in the U.S. bond market three principal components (labelled level, slope and curvature) are sufficient to explain over 95 per cent of the variation in domestic yields (Litterman and Scheinkman, 1991). This stylized fact also holds individually in the four countries examined here (Table 2). Panel A reports the variation in the levels of yields in each country explained by the first k principal components from the cross section of yields. In each country, three "domestic" principal components explain more than 99.9 per cent of the variation in the yield curve. In fact, given that data for very short and long maturities are not available, it can be argued that the four domestic yield curves can be well approximated by only two principal components each (i.e., local level and slope).

Applying a principal component analysis to the cross-section of global yields reveals that more than three components are required to explain the cross-sectional variation in the combined forty interest rates. Panel B of Table 2 shows that eight principal components are needed to explain 99.9 per cent of the variation. The root-mean-squared-pricing-errors (RMSPE) from fitted values of a regression of the yield levels on k principal components are given in Panel C of Table 2. Two domestic principal components in each country deliver RMSPE close to 10 basis points in each of the four countries. To obtain a similar RMSPE we need to use the first eight global principal components (i.e., the same total number of components).

The finding that the same number of principal components are required in both the global and local analysis suggests that some of the components obtained from the former analysis might not be "truly global". Interpreting principal components as global factors can be difficult. Figure 1 plots the loadings of the eight global principal components. If we apply the "global" label to those components that have a similar loading pattern across all four countries, then only the first and fourth principal components qualify as global. The first principal component may be defined as a "global level factor" component since its loadings are constant across maturities and across all four countries. The loadings of the fourth principal component ("global slope factor") are upward sloping for all four

[9] We use OECD PPP-adjusted measures of GDP in 2000 to compute the corresponding weights. Our results are robust to rebalancing the weights every year.

countries. An increase in this component reduces short-term yields and while increasing long-term ones in each country. As all of the other components have loadings that differ across countries or regions, we label them as "local" components.

Perignon, Smith and Villa (2007) discuss the difficulty in identifying principal components obtained from multi-country data as global factors. They note that the objective of principal component analysis is to "extract factors that maximize the explained variance, but not necessarily factors that are common across countries" (Perignon, Smith and Villa (2007), page 286). They advocate using inter-battery factor analysis (IBFA) to extract global factors from international term structure data. The IBFA extracts the true global factor by allowing for the presence of both global and local factors. In the appendix, we show how to use the EM algorithm to help estimate a multi-country version of the IBFA. We can then compare the principal components that we have labelled as global to the global IBFA factors.

The results confirm that the first and fourth principal components are indeed global factors. Panel A of Figure 2 plots the first global IBFA factor along with the first global principal component (i.e., global level).[10] Note that both variables follow each other tightly with a correlation of 0.94. The global level factor is correlated with a proxy measure of global expected inflation (the one-year ahead expectation of U.S. annual inflation from the Survey of Professional Forecasters of the Federal Reserve Bank of Philadelphia). The quarterly correlation between inflation expectations and the IBFA global level factor (first principal component) is 0.90 (0.95). The factors have also consistently trended down slowly since the early 80s. These results match those obtained from the analysis of the U.S. yield curve in Rudebusch and Wu (2008).

Panel B of Figure 2 displays the estimated fourth principal component ("global slope") along with the estimated second global IBFA factor. Again, both variables are strongly correlated (correlation coefficient of 0.84) which allows us to label the fourth principal component as the global slope factor. Panel B also displays NBER recession dates. Similar to the U.S. findings of Rudebusch and Wu (2008), our global slope factor is countercyclical: domestic yield curves steepen during recessions, and flatten during expansions. Similarly, the slope factor usually reaches its minimum level before the start of the recession.

The IBFA analysis thus confirms our labels of global level and global slope for the first and fourth principal components. We note that the combined two factors account for a total of 92.5 per cent of the variation in the international cross-section of bond yields. The global level factor is persistent with a monthly (yearly) autocorrelation of 0.99 (0.94). The global slope factor is less persistent, yet the monthly (yearly) autocorrelation is still high, 0.97 (0.50).

Below, we will use the first eight principal components as our bond market factors to estimate the risk-neutral dynamics of the international term structures. We will also show that the two global factors – global level and global slope – are the only factors with significant risk premia.

[10]Both factors have been rescaled to have unit variance.

2.4 Unspanned risks

One of our main goals is to explore the effect of macroeconomic variables on the market prices of bond and foreign exchange rate risk. In this section, we show that there is a large portion of the variation in macroeconomic variables and exchange rates that is not spanned by the variation in the cross-section of international interest rates.

The first two columns of Table 3 present R^2 statistics from projections of macroeconomic variables and annual rates of depreciation on the eight principal components of interest rates from Table 2. The relatively low values of the statistics indicate that there are economically large fractions of the variation in macroeconomic variables and exchange rates that are not spanned by variation in the cross section of global interest rates. For example, the projection of the global growth proxy on the eight bond market factors delivers an R^2 of 22.3 per cent. Very little is gained in terms of variance explained when we add additional principal components to the regression. A regression of the global inflation proxy on the eight yield factors gives a much larger R^2 of 75.95 per cent. Yet, as noted by JPS, this large R^2 should be taken with caution given the very persistent behavior in inflation and the level of the yield curve.[11] A similar picture can be obtained by looking at domestic measures of inflation and growth for each one of the countries. For example, the projection of the U.S. growth (inflation) proxy on the eight bond market factors delivers an R^2 of 19.42 per cent (71.23 per cent).

We also find that there are economically large fractions of variation in exchange rate movements that are not spanned by variation in international yield curves. Projecting the annual rate of depreciation of the British Pound on the eight bond market factors results in an R^2 of 21.63 per cent. The percentage of variation in the annual rate of depreciation of the German Mark and the Canadian Dollar are 41.22 per cent and 17.30 per cent, respectively.

In addition, the analysis of the last two columns of Table 3 reveals that there is substantial variation in exchange rates that is not unspanned by the global cross-section of interest rates nor macroeconomic variables. Projecting the annual rate of depreciation of the British Pound onto the eight bond market factors and all macroeconomic variables results in an R^2 of 43.82 per cent. A similar exercise for the annual rate of depreciation of the Euro and the Canadian Dollar deliver R^2s of 55.84 per cent and 36.47 per cent.

Thus, we conclude that it is important to allow for variation in the macroeconomic variables that is unspanned by the international cross-section of bond yields, and for variation in exchange rates that is orthogonal to both interest rates and macroeconomic variables.

3 Asset pricing model

3.1 General setup

We describe the state of the global economy by a set of K state variables (or pricing factors). Only the first set of $F < K$ factors, denoted by $\mathbf{f}_t$, are needed to adequately

[11] We note that a similar test using U.S. data over a longer time period shows inflation to be an unspanned risk (see Duffee (2012)).

represent the correlation structure of bond yields. For this reason, we also assume that short rates in each country are affine functions of $\mathbf{f}_t$ only:

$$r_{j,t} = \delta_j^{(0)} + \boldsymbol{\delta}_j^{(1)\prime}\mathbf{f}_t, \qquad j = \$, 1, \ldots, J, \tag{4}$$

which can be represented in compact form as $\mathbf{r}_t = \boldsymbol{\Delta}^{(0)} + \boldsymbol{\Delta}^{(1)}\mathbf{f}_t$, where $\mathbf{r}_t = (r_{\$,t}, r_{1,t}, \ldots r_{J,t})'$, $\boldsymbol{\Delta}^{(0)} = (\delta_{\$}^{(0)}, \delta_1^{(0)}, \ldots, \delta_J^{(0)})'$, and $\boldsymbol{\Delta}^{(1)} = (\boldsymbol{\delta}_{\$}^{(1)}, \boldsymbol{\delta}_1^{(1)}, \ldots, \boldsymbol{\delta}_J^{(1)})'$. For the moment, we remain agnostic as to the nature of these "bond" state variables, as we will discuss our choice of pricing factors in section 4.

In addition, we assume that there are M pricing factors, denoted $\mathbf{m}_t$, that are related to growth, g_{jt}, and inflation, π_{jt}, in the U.S. and each of the J other countries:

$$\mathbf{m}_t = (g_{\$t}, g_{1t}, \ldots, g_{Jt}, \pi_{\$t}, \pi_{1t}, \ldots, \pi_{Jt})'.$$

Finally, we assume that the last J state variables are the rates of depreciation of the J currencies against the U.S. dollar $\Delta\mathbf{s}_t = (\Delta s_{1,t}, \ldots, \Delta s_{J,t})'$ with $\Delta s_{j,t} \equiv s_{j,t} - s_{j,t-1}$, and $s_{j,t}$ is the (log) U.S. dollar price of a unit of foreign currency j.

Collecting all $K = F + M + J$ pricing factors in vector $\mathbf{x}_t$:

$$\mathbf{x}_t = (\mathbf{f}_t', \mathbf{m}_t', \Delta\mathbf{s}_t')',$$

we assume that $\mathbf{x}_t$ follows a VAR(1) process under the physical measure, P, with Gaussian innovations:

$$\mathbf{x}_{t+1} = \boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t + \mathbf{v}_{t+1}, \tag{5}$$

where $\mathbf{v}_t \sim iid\ N(0, \boldsymbol{\Sigma})$.

The model is completed by specifying the U.S. dollar stochastic discount factor (SDF) to be exponentially affine in $\mathbf{x}_t$ (e.g., Ang and Piazzesi, 2003):

$$\xi_{\$,t+1} = \exp\left(-r_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1}\right), \tag{6}$$

with prices of risk given by $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}\mathbf{x}_t$. This (strictly positive) SDF, $\xi_{\$,t+1}$, can be used to price zero-coupon bonds using the following recursive relation:

$$P_{\$,t}^{(n)} = E_t\left[\xi_{\$,t+1}P_{\$,t+1}^{(n-1)}\right], \tag{7}$$

where $P_{\$,t}^{(n)}$ is the price of a zero-coupon bond of maturity n periods at time t. Similarly, it is possible to show that solving equation (7) is equivalent to solving the following equation:

$$P_{\$,t}^{(n)} = E_t^Q\left[\exp\left(-\sum_{i=0}^{n-1} r_{\$,t+i}\right)\right],$$

where E_t^Q denotes the expectation under the risk-neutral probability measure, Q, for the "numeraire currency". Under the risk-neutral probability measure, the dynamics of the state vector $\mathbf{x}_t$ are characterized by the following VAR(1) process:

$$\begin{pmatrix} \mathbf{f}_{t+1} \\ \mathbf{m}_{t+1} \\ \Delta\mathbf{s}_{t+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1^Q \\ \boldsymbol{\mu}_2^Q \\ \boldsymbol{\mu}_3^Q \end{pmatrix} + \begin{pmatrix} \boldsymbol{\Phi}_{11}^Q & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\Phi}_{21}^Q & \boldsymbol{\Phi}_{22}^Q & \boldsymbol{\Phi}_{23}^Q \\ \boldsymbol{\Phi}_{31}^Q & \boldsymbol{\Phi}_{32}^Q & \boldsymbol{\Phi}_{33}^Q \end{pmatrix} \begin{pmatrix} \mathbf{f}_t \\ \mathbf{m}_t \\ \Delta\mathbf{s}_t \end{pmatrix} + \begin{pmatrix} \mathbf{v}_{1,t+1}^Q \\ \mathbf{v}_{2,t+1}^Q \\ \mathbf{v}_{3,t+1}^Q \end{pmatrix}, \tag{8}$$

which can be written in compact form as $\mathbf{x}_{t+1} = \boldsymbol{\mu}^Q + \boldsymbol{\Phi}^Q \mathbf{x}_t + \mathbf{v}_{t+1}^Q$, with $\mathbf{v}_t^Q \sim iid\ N(0, \boldsymbol{\Sigma})$, and

$$\begin{aligned} \boldsymbol{\mu}^Q &= \boldsymbol{\mu} - \boldsymbol{\lambda}_0, \\ \boldsymbol{\Phi}^Q &= \boldsymbol{\Phi} - \boldsymbol{\lambda}. \end{aligned}$$

In order to guarantee that macroeconomic variables, $\mathbf{m}_t$, and the rates of depreciation, $\Delta\mathbf{s}_t$ are not spanned by bond yields, we have imposed two additional restrictions. First, only the bond yield factors drive the short rates in (4). Second, we set the matrix $\boldsymbol{\Phi}_{12}^Q$ and $\boldsymbol{\Phi}_{13}^Q$ (the center and right, upper blocks of the autocorrelation matrix $\boldsymbol{\Phi}^Q$) to zero. Absent these two assumptions, no-arbitrage pricing would imply that bond yields would be affine functions of all $\mathbf{f}_t$, $\mathbf{m}_t$ and $\Delta\mathbf{s}_t$ (cf equations 9 and 12 below). Thus, by inverting the pricing model, it would be possible to recover macro variables and exchange rates from the information contained in yield curves alone and the R^2 statistics obtained in the previous section would equal 1.00. However, our no-spanning assumptions imply that neither $\boldsymbol{\mu}_2^Q$ nor $\boldsymbol{\Phi}_{2\bullet}^Q$ can be identified since they affect neither the prices of the bonds nor their risk premia. The matrices $\boldsymbol{\mu}_3^Q$ and $\boldsymbol{\Phi}_{3\bullet}^Q$ are identified by the absence of arbitrage in the foreign exchange market; i.e., under the risk neutral measure, uncovered interest parity must hold (see appendix).

Solving (7), we find that the continuously compounded yield on an n-period zero coupon bond at time t, $y_{\$,t}^{(n)}$, is given by

$$y_{\$,t}^{(n)} = a_{\$}^{(n)} + \mathbf{b}_{\$}^{(n)\prime} \mathbf{f}_t, \tag{9}$$

where $a_{\$}^{(n)} = -A_{\$}^{(n)}/n$ and $\mathbf{b}_{\$}^{(n)} = -\mathbf{B}_{\$}^{(n)}/n$, and $A_{\$}^{(n)}$ and $\mathbf{B}_{\$}^{(n)}$ satisfy a set of recursive relations (see appendix).

3.2 Stochastic discount factors and exchange rates

By a similar no-arbitrage argument we can postulate the existence of a country j SDF, $\xi_{j,t+1}$, that prices any traded asset denominated in the corresponding currency. We show in the appendix that, when the rate of depreciation is affine in the set of pricing factors (which, in our case is trivially satisfied given that $\Delta s_{j,t+1}$ is itself a pricing factor), the law of one price implies that the rate of depreciation, the numeraire SDF and country j SDF must satisfy the following relation:

$$\Delta s_{j,t+1} = \log \xi_{j,t+1} - \log \xi_{\$,t+1}. \tag{10}$$

Thus, the law of one price tells us that one of the numeraire SDF, the country j SDF and the rate of depreciation of the currency j is redundant and can be constructed from the other two.

When the rate of depreciation is not affine in the factors, an additional assumption of market completeness is needed for equation (10) to be a sufficient and necessary condition for exchange rate determination (Backus, Foresi and Telmer, 2001). In an incomplete markets setting, Brandt and Santa-Clara (2002) introduce an exchange rate factor which is orthogonal to both interest rates and the SDFs in order to match the high degree of exchange rate volatility. Following Anderson, Hammond and Ramezani (2010), we show in the appendix that this approach is not compatible with our assumption of affine rates of depreciation. By restricting the short rates to be functions of only the bond factors in equation (4), and by setting both $\mathbf{\Phi}_{12}^{Q}$ and $\mathbf{\Phi}_{13}^{Q}$ to zero, we are able to introduce variation in exchange rates that is independent of that in macro variables and bond yields.

As in Diez de los Rios (2010), we use (10) to construct a process for the country j SDF implied by our model. Substituting the law of motion for the rate of depreciation in (5) and the domestic SDF in (6) into (10), and imposing uncovered interest parity under the risk-neutral measure, yields the country j SDF with the same form as (6):

$$\xi_{j,t+1} = \exp\left(-r_{j,t} - \frac{1}{2}\boldsymbol{\lambda}_t^{(j)\prime}\mathbf{\Sigma}^{-1}\boldsymbol{\lambda}_t^{(j)} - \boldsymbol{\lambda}_t^{(j)\prime}\mathbf{\Sigma}^{-1}\mathbf{v}_{t+1}\right), \tag{11}$$

with a country j price of risk $\boldsymbol{\lambda}_t^{(j)} = \boldsymbol{\lambda}_t - \mathbf{\Sigma}\mathbf{e}_{F+M+j}$ that is also affine in $\mathbf{x}_t$.[12] Thus, the continuously compounded yield on a foreign n-period zero coupon bond at time t, $y_{j,t}^{(n)}$, is given by:

$$y_{j,t}^{(n)} = a_j^{(n)} + \mathbf{b}_j^{(n)\prime}\mathbf{f}_t, \tag{12}$$

where $a_j^{(n)} = -A_j^{(n)}/n$ and $\mathbf{b}_j^{(n)} = -\mathbf{B}_j^{(n)}/n$, and the scalar $A_j^{(n)}$ and vector $\mathbf{B}_j^{(n)\prime}$ satisfy a set of recursive relations similar to those for the numeraire country.

3.3 Expected returns

The model yields expected holding period returns on the bonds for each country that are affine in the pricing factors. In particular, it is possible to show that the one-year U.S. dollar excess return for holding an n-period zero-coupon bond denominated in U.S. dollars is given by:

$$rx_{\$,t+1}^{(n)} = -\frac{1}{2}\mathbf{B}_{\$}^{(n-1)\prime}\mathbf{\Sigma}_{11}\mathbf{B}_{\$}^{(n-1)} + \mathbf{B}_{\$}^{(n-1)\prime}(\boldsymbol{\lambda}_{10} + \boldsymbol{\lambda}_{11}\mathbf{f}_t + \boldsymbol{\lambda}_{12}\mathbf{m}_t + \boldsymbol{\lambda}_{13}\mathbf{\Delta s}_t + \mathbf{v}_{1,t+1}). \tag{13}$$

By taking expectations, we notice that domestic bond risk premia have three terms: (i) a Jensen's inequality term; (ii) a constant risk premium; and, (iii) a time-varying risk premium where time variation is governed by the parameters in matrix $\boldsymbol{\lambda}_{1\bullet}$. Note that, while macro variables $\mathbf{m}_t$ and exchange rates $\mathbf{s}_t$ do not affect yields, they may help explain time-variation in risk premia.

Similarly, we can compute the U.S. dollar excess return for holding an n-period zero-coupon bond denominated in currency j and hedging the foreign exchange rate risk as:

$$rx_{j,t+1}^{(n)} = -\frac{1}{2}\mathbf{B}_{j}^{(n-1)\prime}\mathbf{\Sigma}_{11}\mathbf{B}_{j}^{(n-1)} + \mathbf{B}_{j}^{(n-1)\prime}(\boldsymbol{\lambda}_{10} - \mathbf{\Sigma}_{13}\mathbf{e}_j + \boldsymbol{\lambda}_{11}\mathbf{f}_t + \boldsymbol{\lambda}_{12}\mathbf{m}_t + \boldsymbol{\lambda}_{13}\mathbf{\Delta s}_t + \mathbf{v}_{1,t+1}). \tag{14}$$

As in the domestic case, foreign bond risk premia have three terms: (i) a Jensen's inequality term; (ii) a constant risk premium; and, (iii) a time-varying risk premium governed by the parameters in matrix $\boldsymbol{\lambda}_{1\bullet}$.

[12] If we denote country j price of risk by $\boldsymbol{\lambda}_t^{(j)} = \boldsymbol{\lambda}_0^{(j)} + \boldsymbol{\lambda}^{(j)}\mathbf{x}_t$, we have that $\boldsymbol{\lambda}_0^{(j)} = \boldsymbol{\lambda}_0 - \mathbf{\Sigma}\mathbf{e}_{F+M+j}$ and $\boldsymbol{\lambda}^{(j)} = \boldsymbol{\lambda}$. Thus, the dynamics of the state vector $\mathbf{x}_t$ under the country j's risk neutral measure will be characterized by a VAR(1) model with constant $\boldsymbol{\mu}^{Q_j} = \boldsymbol{\mu}^{Q} + \mathbf{\Sigma}\mathbf{e}_{F+M+j}$, and autocorrelation matrix $\mathbf{\Phi}^{Q_j} = \mathbf{\Phi}^{Q}$.

Finally, by substituting the particular forms of the domestic and foreign SDFs in equations (6) and (11) into (10), we can also compute the excess return earned by a domestic investor for holding a one-year zero-coupon bond from country j:

$$sx_{j,t+1} = -\frac{1}{2}\mathbf{e}_j'\mathbf{\Sigma}_{33}\mathbf{e}_j + \mathbf{e}_j'(\boldsymbol{\lambda}_{30} + \boldsymbol{\lambda}_{31}\mathbf{f}_t + \boldsymbol{\lambda}_{32}\mathbf{m}_t + \boldsymbol{\lambda}_{33}\mathbf{\Delta s}_t + \mathbf{v}_{3,t+1}). \tag{15}$$

As with the case of bond risk premia, by taking expectations, we can see that foreign exchange expected returns have three terms: (i) a Jensen's inequality term, (ii) a constant risk premium, and (iii) a time-varying risk premium governed by the matrix $\boldsymbol{\lambda}_{3\bullet}$.

We note that bond risk premia contain sufficient information to identify $\boldsymbol{\lambda}_{10}$ and $\boldsymbol{\lambda}_{1\bullet}$ while currency risk premia identify $\boldsymbol{\lambda}_{30}$ and $\boldsymbol{\lambda}_{3\bullet}$. As noted earlier, there is no information in either bond or foreign exchange premia to identify the price of macroeconomic risk, that is, $\boldsymbol{\lambda}_{20}$ and $\boldsymbol{\lambda}_{2\bullet}$.

4 Estimation

The estimation of both domestic and international dynamic term structure models is challenging because their (quasi) log-likelihood functions have a large number of local maxima. In these models, risk factors are usually latent with the result that estimates of the parameters governing the historical distribution, P, usually depend on those governing the risk-neutral distribution, Q, (i.e., one has either to invert the model to obtain the fitted states or to filter the risk factors out). This has restricted the literature on international term structure models to focusing on two-country models while limiting the number of state variables considered. To overcome this problem, we follow JSZ in working with "bond" state variables that are linear combinations (i.e., portfolios) of the yields themselves, $\mathbf{f}_t = \mathbf{P}'\mathbf{y}_t$, where $\mathbf{P}$ is a full-rank matrix of weights. In particular, we choose these weights in such a way that $\mathbf{f}_t$ are the first F principal components of the international cross-section of yields.

However, when choosing state variables that are linear combinations (portfolios) of the yields, one has to guarantee that the model is self-consistent in the sense of Cochrane and Piazzesi (2005): the state variables that come out of the model need to be the same as the state variables that we started with. In a one-country world, JSZ show how to translate these self-consistency restrictions into restrictions on the parameters that govern the dynamic evolution of the state variables under the risk neutral measure. We adapt their approach to a multi-country framework. In particular, we show that a self-consistent multi-country term structure model is observationally equivalent to a canonical model with latent state variables and restrictions on both the parameters that govern the dynamic evolution of the state variables under the risk neutral measure and the loadings of the short-rates across the different countries. We collect such result in Lemma 1 and Proposition 2.

Lemma 1 *The generic representation of a multi-country term structure model in equations (4), (5) and (8) is observationally equivalent to a model where:* ***(1)*** *the short rates are linear in a set of latent "bond" factors* $\mathbf{z}_t$

$$\mathbf{r}_t = \mathbf{\Gamma}^{(1)}\mathbf{z}_t, \tag{16}$$

*where $\mathbf{\Gamma}^{(1)}$ is a matrix that stacks the short-rate loadings on each of the factors and satisfies $\mathbf{1}'_{J+1}\mathbf{\Gamma}^{(1)}= \mathbf{1}'_F$, where $\mathbf{1}_n$ is a n-dimensional vector of ones (that is, the sum of each of the columns of $\mathbf{\Gamma}^{(1)}$ is equal to one); **(2)** the joint dynamic evolution of the latent bond factors, macroeconomic variables and exchange rates, $\widetilde{\mathbf{x}}_t = (\mathbf{z}'_t, \mathbf{m}'_t, \Delta\mathbf{s}'_t)'$, under the risk neutral measure is given by the following VAR(1) process:*

$$\begin{pmatrix} \mathbf{z}_{t+1} \\ \mathbf{m}_{t+1} \\ \Delta\mathbf{s}_{t+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\theta}_1^Q \\ \boldsymbol{\theta}_2^Q \\ \boldsymbol{\theta}_3^Q \end{pmatrix} + \begin{pmatrix} \mathbf{\Psi}_{11}^Q & \mathbf{0} & \mathbf{0} \\ \mathbf{\Psi}_{21}^Q & \mathbf{\Psi}_{22}^Q & \mathbf{\Psi}_{23}^Q \\ \mathbf{\Psi}_{31}^Q & \mathbf{\Psi}_{32}^Q & \mathbf{\Psi}_{33}^Q \end{pmatrix} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{m}_t \\ \Delta\mathbf{s}_t \end{pmatrix} + \begin{pmatrix} \mathbf{u}_{1,t+1}^Q \\ \mathbf{u}_{2,t+1}^Q \\ \mathbf{u}_{3,t+1}^Q \end{pmatrix}, \tag{17}$$

*which can be represented in compact form as $\widetilde{\mathbf{x}}_{t+1} = \boldsymbol{\theta}^Q + \mathbf{\Psi}^Q\widetilde{\mathbf{x}}_t + \mathbf{u}_{t+1}^Q$, where $\mathbf{u}_t^Q \sim iid$ $N(0, \mathbf{\Omega})$, $\boldsymbol{\theta}_1^Q = (\mathbf{k}_\infty^{Q\prime}, \mathbf{0}'_{F-J-1})'$ is a vector where the first $J+1$ elements are different from zero, the matrix $\mathbf{\Psi}_{11}^Q$ is in ordered real Jordan form, and $\boldsymbol{\theta}_3^Q$ and $\mathbf{\Psi}_{3\bullet}^Q$ satisfy restrictions analogous to those in the appendix that guarantee that uncovered interest parity holds under the risk neutral measure; and **(3)** $\widetilde{\mathbf{x}}_t$ follows an unrestricted VAR(1) process under the historical measure: $\widetilde{\mathbf{x}}_{t+1} = \boldsymbol{\theta} + \mathbf{\Psi}\widetilde{\mathbf{x}}_t + \mathbf{u}_{t+1}$, where $\mathbf{u}_t \sim iid$ $N(0, \mathbf{\Omega})$.*

Remark 1 *When the eigenvalues in $\mathbf{\Psi}_{11}^Q$ are real and distinct, $\mathbf{\Psi}_{11}^Q$ is a diagonal matrix. Furthermore, as noted by Hamilton and Wu (2012), the elements of $\mathbf{\Psi}_{11}^Q$ have to be in descending order, $\psi_{11,1}^Q > \psi_{11,2}^Q > \ldots \psi_{11,F}^Q$, in order to have a globally identified structure.*

Remark 2 *The representation in Lemma 1 nests the models proposed in Joslin and Graveline (2010) and Jotikasthira, Le and Lundblad (2010) under appropriate zero restrictions on $\mathbf{\Gamma}^{(1)}$.*

The dynamic term structure model given in this lemma is a multi-country version of the canonical model in Proposition 1 in JSZ and extended to the case of unspanned macro risks in JPS. Note that such a model implies that yields on domestic and foreign zero coupon bonds are affine in $\mathbf{z}_t$:

$$\mathbf{y}_t = \mathbf{a}_z + \mathbf{b}_z\mathbf{z}_t. \tag{18}$$

Thus, state variables that are linear combinations of the yields can simply be understood as an affine (invariant) transformation of the latent factors $\mathbf{z}_t$:

Proposition 2 *The multi-country term structure model given by equations (4), (5) and (8), with state variables that are linear combinations of yields, $\mathbf{f}_t = \mathbf{P}'\mathbf{y}_t$, is self-consistent when*

$$\begin{aligned} \mathbf{\Delta}^{(1)} &= \mathbf{\Gamma}^{(1)}\mathbf{D}^{-1} \\ \mathbf{\Delta}^{(0)} &= -\mathbf{\Delta}^{(1)}\mathbf{c} \\ \mathbf{\Phi}_{11}^Q &= \mathbf{D}\mathbf{\Psi}_{11}^Q\mathbf{D}^{-1} \\ \boldsymbol{\mu}_1^Q &= (\mathbf{I} - \mathbf{\Phi}_{11}^Q)\mathbf{c} + \mathbf{D}\boldsymbol{\theta}_1^Q \end{aligned}$$

where $\mathbf{c} = \mathbf{P}'\mathbf{a}_z$, $\mathbf{D} = \mathbf{P}'\mathbf{b}_z$ and $\mathbf{a}_z$, $\mathbf{b}_z$ are implicitly defined in equation (18). The parameters under the physical measure remain unrestricted.

A distinctive feature of our multi-country model with observable factors is that there is a separation between the parameters driving the state variables under the historical distribution and those in the risk-neutral distribution. This greatly simplifies the estimation of our model as: (1) the cross-section of bond prices is fully determined by the risk-neutral dynamics ($\boldsymbol{\mu}_1^Q$, $\boldsymbol{\Phi}_{11}^Q$ and $\boldsymbol{\Sigma}$) and the parameters of the short rates ($\boldsymbol{\Delta}^{(0)}$ and $\boldsymbol{\Delta}^{(1)}$); while, (2) the time-series properties of the state factors are determined by the parameters in $\boldsymbol{\mu}$ and $\boldsymbol{\Phi}$ only. Using this separation, we can estimate all of the parameters of the model in three steps. First, we estimate the parameters of the risk-neutral dynamics that provide the best match for the cross-section of international bond yields. Second, we exploit the fact that risk premia in our model are affine in the state variables to obtain estimates of the prices of risk. Finally, we recover the parameters under the historical distribution using our estimates of the risk-neutral measure and prices of risk parameters. We describe each step in turn.

4.1 Step 1: Fitting yields

We start by estimating the parameters of the risk-neutral distribution using the cross-section of international bond yields. We follow Cochrane and Piazzesi (2008) in estimating $\boldsymbol{\Sigma}$ from the innovation covariance matrix of an OLS estimate of the unrestricted VAR(1) dynamics in equation (5). We are then able to estimate the parameters directly by minimizing the sum (across maturities, countries, and time) of the squared differences between model predictions and actual yields:

$$\min_{\boldsymbol{\mu}_1^Q, \boldsymbol{\Phi}_{11}^Q, \boldsymbol{\Delta}^{(0)}, \boldsymbol{\Delta}^{(1)}} \sum_{n=1}^{N} \sum_{j=1}^{J+1} \sum_{t=1}^{T} (y_{j,t}^{(n)} - a_j^{(n)} - \mathbf{b}_j^{(n)\prime} \mathbf{f}_t)^2. \tag{19}$$

subject to the self-consistency restrictions in Proposition 2.[13] Once the parameters that govern the dynamics of bond factors under the risk neutral measure have been estimated, we can then recover the parameters that govern the dynamics of exchange rates under the same measure as uncovered interest parity holds under Q.

As the yield curve does not span the macroeconomic risks, $\boldsymbol{\mu}_2^Q$ and $\boldsymbol{\Phi}_{2\bullet}^Q$ cannot be identified from the cross-section of international bond yields. Thus, our estimates of the risk-neutral parameters will be the same for a "yields-only" model and a model that includes macroeconomic factors to help explain the evolution of bond risk premia. Similarly, these estimates are invariant to the restrictions that we impose on the prices of risk below.

4.2 Step 2: Estimating the prices of risk

Once we have obtained estimates of the parameters governing the dynamics of the pricing factors under the risk-neutral measure, we can estimate the parameters driving the prices of risk ($\boldsymbol{\lambda}_0$ and $\boldsymbol{\lambda}$). As noted in section 3.3, there is a separation between the parameters driving the prices of bond risk ($\boldsymbol{\lambda}_{10}$ and $\boldsymbol{\lambda}_{1\bullet}$) and those driving exchange rate risk ($\boldsymbol{\lambda}_{30}$

[13] As in Christensen, Diebold and Rudebusch (2011), we set the largest eigenvalue of $\boldsymbol{\Phi}_{11}^Q = 1.00$ in order to replicate the level factor that characterizes the international cross-section of interest rates. See appendix.

and $\boldsymbol{\lambda}_{3\bullet}$). We could thus obtain estimates of the parameters driving the prices of bond risk from OLS regressions on the bond pricing factors:

$$\mathbf{f}_{t+1} - \left(\widehat{\boldsymbol{\theta}}_1^Q + \widehat{\boldsymbol{\Phi}}_{11}^Q \mathbf{f}_t\right) = \boldsymbol{\lambda}_{10} + \boldsymbol{\lambda}_{11}\mathbf{f}_t + \boldsymbol{\lambda}_{12}\mathbf{m}_t + \boldsymbol{\lambda}_{13}\Delta\mathbf{s}_t + \mathbf{v}_{1,t+1}, \tag{20}$$

where $\widehat{\boldsymbol{\theta}}_1^Q$ and $\widehat{\boldsymbol{\Phi}}_{11}^Q$ are estimates of the parameters under the risk-neutral measure obtained in the first step. Similarly, we could obtain estimates of the parameters driving the price of foreign exchange rate risk ($\boldsymbol{\lambda}_{30}$ and $\boldsymbol{\lambda}_{3\bullet}$) from the regressions:

$$\Delta\mathbf{s}_{t+1} - \left(\widehat{\boldsymbol{\theta}}_3^Q + \widehat{\boldsymbol{\Phi}}_{31}^Q \mathbf{f}_t\right) = \boldsymbol{\lambda}_{30} + \boldsymbol{\lambda}_{31}\mathbf{f}_t + \boldsymbol{\lambda}_{32}\mathbf{m}_t + \boldsymbol{\lambda}_{33}\Delta\mathbf{s}_t + \mathbf{v}_{3,t+1}. \tag{21}$$

However, there are three reasons to impose restrictions on the prices of risk. The first concerns the trade-off between model mis-specification and sampling uncertainty. As noted by Cochrane and Piazzesi (2008), the risk-neutral distribution can provide a lot of information about the time-series dynamics of the yields. For example, if the price of risk were zero (i.e., agents were risk-neutral), both physical and risk-neutral dynamics would coincide and we could obtain estimates of the parameters driving the time-series process of yields exclusively from the cross-section of interest rates. Since the risk-neutral dynamics can be measured with great precision (in our case with RMSPE of less than 10 basis points), one could reduce the sampling uncertainty by following this approach. We will show the results for this model below and label it the "risk-neutral model."

On the other hand, when the prices of risk are completely unrestricted, no-arbitrage restrictions are irrelevant for the conditional distribution of yields under the physical measure and thus the cross-section of bond yields does not contain any information about the time-series properties of interest rates (see JSZ). In this case, it can be shown that the estimates of the physical dynamic parameters, $\boldsymbol{\mu}$ and $\boldsymbol{\Phi}$, coincide with the OLS estimates of an unrestricted VAR(1) process for $\mathbf{x}_t$. We will refer to this model in the subsequent sections as the "unrestricted model." Our approach of imposing restrictions on the prices of risk can be understood as a trade-off between these two extreme cases.

The second reason concerns the estimated persistence of the data. When the prices of risk are completely unrestricted, the largest eigenvalue of the physical measure $\boldsymbol{\Phi}$ estimated from the VAR(1) representation in equation (5) is usually less than 1.00 with the result that expected future bond yields beyond ten years are almost constant.[14] However, the existence of a level factor in the cross-section of interest rates implies a very persistent process for bond yields under the risk-neutral measure. The largest eigenvalue of $\boldsymbol{\Phi}_{11}^Q$ thus tends to be close or equal to one. By imposing restrictions on the prices of risk, we will be effectively pulling the largest eigenvalue of $\boldsymbol{\Phi}$ closer to that of $\boldsymbol{\Phi}_{11}^Q$ so that the physical time-series can inherit more of the high persistence that exists under the risk-neutral measure.

The third reason for imposing restrictions on the prices of risk is related to the effects of over-specification on the Sharpe ratios implied by the model. As noted by Duffee

[14]Problems with measuring the persistence of the term structure physical dynamics given the short data samples available have been noted by Ball and Torus (1996), Bekaert, Hodrick and Marshall (1997), Kim and Orphanides (2005), Cochrane and Piazzesi (2008), Duffee and Stanton (2008), Bauer (2010), Bauer, Rudebusch and Wu (2011), and JPS.

(2010), over-specification might deliver implausibly high Sharpe ratios. Indeed, we find below that the magnitude and volatility of the conditional Sharpe ratios for bond and currency portfolios implied by a model with unrestricted prices of risk are unrealistically high.

For these reasons we adopt the following economic restrictions on our estimates of the $\boldsymbol{\lambda}$ parameters.

1. **Global asset pricing**: Under the assumption of completely integrated international financial markets, investors diversify away their exposures to local factors with the result that only global risks command risk premia. Assets with the same exposures to the global risks will have identical expected returns regardless of their country of origin. Global factors in developed-country international bond returns are documented by Ilmanen (1995), Harvey, Solnik and Zhou (2002), Driessen, Melenberg and Nijman (2003), Perignon, Smith and Villa (2007), Bekaert and Wang (2009), Dahlquist and Hasseltoft (2011), and Hellerstein (2011).[15]

   This assumption implies that bond market expected returns will be driven by compensation for shocks to the global level and global slope factors only. This imposes a large number of zero restrictions in the $\boldsymbol{\lambda}_{11}$ and $\boldsymbol{\lambda}_{12}$ matrices. The first and fourth rows of $\boldsymbol{\lambda}_{11}$ and $\boldsymbol{\lambda}_{12}$, which correspond to the compensation for global level and global slope risks, respectively, can take on values different from zero. All other rows are set to zero. We further assume that time variation in the prices of global level and slope risks are driven by global variables only. We thus set the columns of $\boldsymbol{\lambda}_{11}$ that correspond to local bond market factors to zero. In addition, we constrain the columns of $\boldsymbol{\lambda}_{12}$ so that variation in the price of global risks is driven by global growth and global inflation.[16] As there is very little evidence of increased explanatory power when trying to forecast bond holding period returns using the rates of depreciation, we also impose $\boldsymbol{\lambda}_{13} = 0$.

2. **Carry-trade fundamentals**: We have assumed that exchange rate risks are global, so they are priced in equilibrium. There is a large number of theoretical and empirical papers that support this claim (e.g., Adler and Dumas (1983), De Santis and Gerard (1997)). Recently, attention has focused on the time variation in ex-

[15] For a similar results in emerging market bonds see Longstaff, Pan, Pedersen and Singleton (2011). Global asset pricing has long been tested in studies of developed country equity markets as well where one would expect informational asymmetries and other potential frictions to cause a greater degree of segmentation than that found in fixed income markets (e.g., Campbell and Hamao (1992), Harvey, Solnik and Zhou (2002), Bekaert and Wang (2009), Hau (2009), Rangvid, Schmeling and Schrimpf (2010) and Lewis (2011)). While there is mixed evidence of local factors being priced in global equity markets, we maintain the assumption of global asset pricing in the bond markets and revisit the implications of allowing for priced local factors in section 5 below.

[16] This can be achieved via the following restrictions:

$$e_i'\boldsymbol{\lambda}_{12} = \gamma_{ig}\boldsymbol{\omega}^g + \gamma_{i\pi}\boldsymbol{\omega}^\pi, \qquad i = 1, 4$$

where $\boldsymbol{\omega}^g = (\omega_1, \ldots, \omega_J, 0, \ldots, 0)$ and $\boldsymbol{\omega}^\pi = (0, \ldots, 0, \omega_1, \ldots, \omega_J)$ are vectors of (known) weights, ω_j, that when premultiplying by the vector of macroeconomic variables give our measures of global growth and global inflation (i.e. $g_{w,t} = \boldsymbol{\omega}^g \mathbf{m}_t$ and $\pi_{w,t} = \boldsymbol{\omega}^\pi \mathbf{m}_t$); and $\gamma_{i\pi}$ and γ_{ig} are parameters to be estimated.

pected returns on carry trade portfolios and their link to global factors.[17] There is also a recent literature on present value models of exchange rates over short and long horizons (e.g., Groen (2000), Engel and West (2005), Cheung, Chinn and Pascual (2005), Molodtsova and Papell (2009)) where the predictive abilities of relative macroeconomic variables (i.e., the U.S. variable less its foreign counterpart) for the exchange rate are evaluated.

We impose restrictions on the prices of foreign exchange risk to reflect these previous findings. First, we assume that the bond market factors cause time variation in the price of foreign exchange risk via the carry (i.e. the difference between the one-year interest rates of the two countries):

$$\mathbf{e}_j'\boldsymbol{\lambda}_{31} = \beta_{1j}(\boldsymbol{\delta}_{\$}^{(1)} - \boldsymbol{\delta}_j^{(1)}), \qquad j = 1, \ldots, J,$$

where β_{1j} is a parameter to be estimated. Thus, the $\boldsymbol{\lambda}_{31}$ matrix has a reduced rank structure. Second, we constrain $\boldsymbol{\lambda}_{32}$ so that the difference between the U.S. and country j's growth and inflation rates cause time variation in the currency risk premia. In addition, rates of currency depreciation have low autocorrelation so we set $\boldsymbol{\lambda}_{33}$ to zero.

3. **Maximal Sharpe ratios**: Below, we show that the assumptions of global asset pricing and carry-trade fundamentals are not sufficient by themselves to yield plausible Sharpe ratios. We therefore also use shrinkage to restrict both the maximal and average Sharpe ratios implied by the model.[18] We construct a weighted average of the model's estimated prices of risk and a zero price of risk:

$$\begin{aligned} \widehat{\boldsymbol{\lambda}}_{1t}^{s} &= (1-\gamma)\widehat{\boldsymbol{\lambda}}_{1t}, \\ \widehat{\boldsymbol{\lambda}}_{3t}^{s} &= (1-\gamma)\widehat{\boldsymbol{\lambda}}_{3t}, \end{aligned} \tag{22}$$

where $\widehat{\boldsymbol{\lambda}}_{jt}$ is the (OLS) estimate of the prices of bond and foreign exchange risk, and $\widehat{\boldsymbol{\lambda}}_{jt}^{s}$ is the shrinkage estimator. The common shrinkage parameter γ (which lies between zero and one) controls how much we tilt our estimates towards the assumption of risk neutrality. Since the implied Sharpe ratio under risk neutrality is zero, the shrinkage parameter allows us to control the properties of the Sharpe ratios implied by the model.

A model that imposes these three sets of economic restrictions will be referred below as "restricted model."

As noted by Duffee (2012), the literature on the domestic term structure has mainly relied on statistical methods to impose restrictions of the prices of risk (e.g., Duffee (2002), Dai and Singleton (2002), JPS, Christensen, Lopez and Rudebusch (2010), Bauer (2011)).

[17] The carry trade has been examined by Lustig and Verdelhan (2007), Lustig, Roussanov and Verdelhan (2008), Brunnermeier, Nagel and Pedersen (2009), Farhi and Gabaix (2011), Bansal and Shaliastovich (2010), Verdelhan (2010), Ang and Chen (2010), Lustig, Roussanov and Verdelhan (2011), Burnside (2011) and Burnside, Eichenbaum and Rebelo (2011).

[18] Introduced by Stein (1956), shrinkage has been traditionally used to achieve a better trade-off between the bias and the estimation variance components of the mean squared forecast error.

In contrast, in this paper we base the restrictions on the prices of risk, $\boldsymbol{\lambda}$, from economic theory and previous empirical work external to the affine term structure model. We note that there is a separate consumption based asset pricing approach that delivers an affine term structure model (see, for example, Gallmeyer, Hollifield, Palomino and Zin (2007) and Garcia and Luger (2011) along with the discussions in Gurkaynak and Wright (2010) and Duffee (2012)) with explicit restrictions on the prices of risk that arise from the model. Our approach may be viewed as an intermediate step between these two literatures.

Finally, note that, since data is sampled monthly but t is measured in years, (20) and (21), we need to compute standard errors that are robust to the serial correlation that exists in the error terms. The standard errors must also be corrected for the first-stage estimation of the risk-neutral parameters. Additional details on the computation of standard errors can be found in the appendix.

4.3 Step 3: Recovering the parameters under the physical measure

In our last step, we recover the parameters driving the annual dynamics of bond pricing factors and exchange rates under the physical measure using the estimates of the parameters under the risk neutral measure obtained in the first step and the restricted prices of risk estimates obtained in the second step. Specifically,

$$\begin{array}{rclr} \widehat{\boldsymbol{\mu}}_1 & = & \widehat{\boldsymbol{\mu}}_1^Q + \widehat{\boldsymbol{\lambda}}_{10}^s & \widehat{\boldsymbol{\mu}}_3 = \widehat{\boldsymbol{\mu}}_3^Q + \widehat{\boldsymbol{\lambda}}_{30}^s, \\ \widehat{\boldsymbol{\Phi}}_{1\bullet} & = & \widehat{\boldsymbol{\Phi}}_{1\bullet}^Q + \widehat{\boldsymbol{\lambda}}_{1\bullet}^s & \widehat{\boldsymbol{\Phi}}_{3\bullet} = \widehat{\boldsymbol{\Phi}}_{3\bullet}^Q + \widehat{\boldsymbol{\lambda}}_{3\bullet}^s. \end{array} \quad (23)$$

Finally, we estimate the parameters driving macroeconomic factors under the physical measure ($\boldsymbol{\mu}_2$ and $\boldsymbol{\Phi}_{2\bullet}$) from the following regression:

$$\mathbf{m}_{t+1} = \boldsymbol{\mu}_2 + \boldsymbol{\Phi}_{21}\mathbf{f}_t + \boldsymbol{\Phi}_{22}\mathbf{m}_t + \boldsymbol{\Phi}_{23}\Delta\mathbf{s}_t + \mathbf{v}_{2,t+1},$$

subject to two restrictions. First, the U.S. growth and inflation rates, $\mathbf{m}_{\$,t} = (g_{\$,t}, \pi_{\$,t})'$ only depend on past values of the U.S. short-rate and slope of the yield curve (the difference between the ten-year and one-year yield), as well as $\mathbf{m}_{\$,t-1}$. Second, the macroeconomic variables in each of the other j countries, $\mathbf{m}_{j,t} = (g_{j,t}, \pi_{j,t})'$, depend on the past values of that country's own short-rate and slope of the yield curve in addition to the U.S. ones, as well as past values of both $\mathbf{m}_{j,t-1}$ and $\mathbf{m}_{\$,t-1}$. As such $\boldsymbol{\Phi}_{23} = 0$.[19]

5 Empirical Results

5.1 Fitted yields

Table 4 shows the estimates of the factor loadings of short-term interest rates under the canonical representation in Lemma 1. We find that most of the elements in $\boldsymbol{\Gamma}^{(1)}$ are

[19]Note also that given these estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Phi}$ we can potentially update our estimate of the innovation covariance matrix of the VAR(1) dynamics in equation (5) and re-estimate our model again. In practice, such updating makes little difference.

statistically different from zero. In fact, short-rates in different countries have significant coefficients on most of the canonical factors, $\mathbf{z}_t$.

Figure 1 presents both the estimated bond yield loadings implied by the affine term structure model as well as the regression coefficients that one would obtain from projecting bond yields on the first eight principal components (i.e., the loadings from a principal components analysis). The latter coefficients are from a linear factor model that minimizes the sum of the squared differences between model predictions and actual yields in (F.1). They thus provide a natural benchmark to compare the pricing errors implied by our no-arbitrage model. Figure 1 shows that the multi-country term structure model is flexible enough to replicate the shapes of the loadings on individual bond yields obtained from a principal component analysis.

We confirm the model's fit by providing root mean squared pricing errors (RMSPE) and mean absolute pricing errors (MAPE) in Table 5. The column labelled "Affine" provides estimates of the goodness-of-fit measures for the affine term structure model; the column "OLS" gives the results for an unrestricted regression of bond yields on the global principal components; while "Difference" characterizes the difference between the two quantities. The loss from imposing the no-arbitrage conditions is minimal: the difference in pricing errors is less than one basis point at either the country or global level.

5.2 Prices of risk

Estimates of the prices of risk subject to the restrictions of global asset pricing and carry-trade fundamentals are displayed in Table 6.[20] Panel A focuses on the coefficients driving bond risk premia. We find that both global level and global slope risks are priced as both rows contain (statistically significant) non-zero entries. It is interesting to note that the pattern of the signs on the coefficients driving the compensation for global level risk is the same as the one found by JPS for the U.S. term structure. The expected excess return is negatively affected by both the global level and global slope factors, while the coefficients on global growth and inflation are positive (though the coefficient on global growth is not significant). Risk premia on the global level exposures are thus pro-cyclical.

We also find that compensation for slope risk is priced. However, it displays a counter-cyclical pattern as the coefficient on global growth is negative. The fact that inflation does not seem to drive the price of slope risk is also consistent with the results in JPS. The finding that both global level and slope risks are priced is in line with the results in JPS and Duffee (2010) and differs from those in Cochrane and Piazzesi (2008), who find that only level risk is priced in the term structure of U.S. interest rates. Our use of macroeconomic variables may be responsible for finding a significantly priced second factor (as in JPS).

The estimated coefficients driving the foreign exchange risk premia are in Panel B of Table 6. There is a negative sign on the carry factor for all three currencies under consideration, though the estimated coefficient is not significant for the Euro - U.S. dollar exchange rate. The negative coefficients are consistent with the forward premium puzzle: high domestic interest rates relative to those in the U.S. predict an appreciation of the

[20] For space considerations, we do not report estimates of the prices of risk coefficients for the unrestricted model. However, we will show the model's implications for Sharpe ratios and forecasts below.

home currency. The coefficients on the growth rate differential are also negative for the three countries indicating that a country that is growing faster than the U.S. will have a depreciating currency. The coefficients on the inflation differentials are also negative for the Euro and the Canadian dollar, while positive for the British Pound. Both results indicate that foreign exchange risk premia are countercyclical to the U.S. economy. This matches the results in Lustig and Verdelhan (2007), Lustig, Roussanov and Verdelhan (2010) and Sarno, Schneider and Wagner (2011).

5.3 Implied Sharpe ratios

We use conditional Sharpe ratios to provide an economic measure of the degree of over-specification of our models. Previous papers indicate that realistic average and maximal Sharpe ratios for investments in either government bond or foreign exchange markets are below 1.00. Duffee (2010) considers 0.15 to 0.20 as a reasonable benchmark for the maximum unconditional monthly Sharpe ratio for U.S. term structure data (i.e. annual Sharpe ratios between 0.5 and 0.7). Campbell, de Medeiros and Viceira (2010) find average Sharpe ratios of 0.5 for developed country bond portfolios constructed using different measures of currency hedging over the 1975-2005 period. In their sub-period analysis, the ratios may reach levels of approximately 0.8. In the foreign exchange market, Lustig, Roussanov and Verdelhan (2011) examined returns to carry-trade portfolios over the 1983-2009 period. For developed countries, the annualized Sharpe ratio for the high minus low portfolio is 0.61. Brunnermeier, Nagel and Pedersen (2008) find Sharpe ratios of 0.8 or less.

Figure 3 displays the average and maximum (over the sample) of the maximally-obtainable conditional Sharpe ratios for an agent who invests only in bonds, an agent who is restricted to invest only in currencies, and an agent who is allowed to invest in both bonds and currencies. The Sharpe ratios are plotted as functions of the shrinkage parameter (22). The larger the parameter, the greater the reduction in the Sharpe ratio towards risk neutrality (a Sharpe ratio of 0).

The top lines in the top two graphs are the average and maximal (respectively) Sharpe ratios for a portfolio of international bonds obtained from an "unrestricted" model which results from applying OLS to (14). In this unrestricted model, all local and global factors are priced in bond returns. These results show that the unrestricted model is inconsistent with reasonable Sharpe ratios; the average Sharpe ratio is approximately 3.3 while the maximum is close to 7.0 if no shrinkage is applied. A very large shrinkage parameter would be required to restrict the Sharpe ratio to reasonable levels.

On the other hand, the imposition of global asset pricing ("Global Level + Global Slope") on one-year excess returns results in a much flatter Sharpe ratio line. The intercept for the average Sharpe ratio fall to 0.8 while that for the maximal declines to 2.0. Imposing a shrinkage parameter $\gamma = 0.5$ yields average and maximum Sharpe ratios of approximately 0.5 and 1.0, respectively, for investments in bonds.

It is important to realize that there is a large and unrealistic increase in the Sharpe ratios if we allow local factors to be priced. The middle line in each graph ("Domestic Asset Pricing") shows the trade-off if we estimate an individual model for each country, where we allow for non-zero prices of risk for each country's two domestic principal components (i.e., domestic level and slope). (See appendix for details). While there is a reduction in the average and maximal Sharpe ratios from the completely unrestricted model, the reductions do not result in realistic Sharpe ratios. The large maximum Sharpe ratios (e.g. approximately equal to 3.75 with no shrinkage) indicate that there is a lot of variation in the prices of risk of the local factors which may be unrealistically high (i.e., a large degree of in-sample overfitting of the estimated coefficients on the local factor risk premia).

There is a similar effect in the model's implications for the carry-trade fundamental restrictions in the foreign exchange market. The unrestricted model (the top line in each of the graphs in the middle panel) results from applying OLS to (15). Absent any restrictions, the average Sharpe ratio with no shrinkage for a portfolio of investments in the currencies is above 1.6 (maximum near 4.5). Imposing the carry-trade fundamentals reduces the values to approximately 0.7 and 2.0, respectively. Again, imposing a shrinkage parameter of 0.5 yields reasonable results.

The two graphs in the bottom panel show the combined effects of the two economic restrictions on a portfolio of bonds and currencies. A completely unrestricted model continues to yield large average and maximal Sharpe ratios. Using a shrinkage parameter of 0.5 results in an average Sharpe ratio of approximately 0.5. The same parameter imposes maximum Sharpe ratios of 1.0 in the individual asset classes, while the combined Sharpe ratio is approximately 1.5.

Thus, even after the imposition of global asset pricing combined with carry trade fundamentals, some degree of shrinkage is necessary to obtain Sharpe ratios that are consistent with our priors as to what a sensible Sharpe ratio should be. We adopt the value of $\gamma = 0.5$ and use this "restricted" model in our subsequent analysis.

6 Decomposing Forward Curves

6.1 Long-run expectations and survey data

In this section, we analyze long-run expectations of interest rates and exchange rates from the restricted model. We compare the model's forecasts to those from survey data.

6.1.1 Interest rates

The forward interest rates – the interest rate at time t for loans between time $t+n-1$ and $t+n$ – can be computed as

$$f_{j,t}^{(n)} = \log P_{j,t}^{(n-1)} - \log P_{j,t}^{(n)}.$$

The forward interest rate can be decomposed into the expected yield on a one-year bond purchased $n-1$ years from now, $E_t y_{j,t+n-1}^{(1)}$, and a risk premium term, $ftp_{j,t}^{(n)}$, called the forward term premium:

$$f_{j,t}^{(n)} = E_t y_{j,t+n-1}^{(1)} + ftp_{j,t}^{(n)}. \tag{24}$$

An investor buys an n-year bond and holds it to maturity while financing the position by shorting an $(n-1)$-year bond to maturity and then selling a one-year bond from $t+n-1$ to $t+n$. The forward term premium is the expected excess expected return on this portfolio.[21] By definition, the unspanned macroeconomic variables do not affect the (forward) interest rates. However, they will help forecast the two components: future short-term interest rates and forward term premia (with off-setting effects). Thus, to correctly assess the impact of any unspanned variables we need to ensure that the model produces an accurate decomposition. We therefore compare the model's forecast of future interest rates to those from survey data.

We start by analyzing the restricted model's implications for expected future short-term interest rates, the first term in (24). Figure 4 plots the current one-yield yields, $y_{j,t}^{(1)}$, and the expected one-year rate in 10 years, $E_t y_{j,t+10}^{(1)}$, generated by the three different models. The first "risk-neutral model" assumes that the prices of risk are zero so that the dynamics of the state variables are given by (8). Note that the predictions of this model are equal to the implied rates on an "in-10-for-1" forward loan (i.e., a one-year loan initiated in 10 years), $f_{j,t}^{(11)}$, up to a convexity adjustment. The second "unrestricted model" uses empirical estimates of the prices of risk without our economic restrictions (i.e., the model with the Sharpe ratios shown in the top lines in Figure 3). The forecasts from this model are equivalent to those from an unrestricted VAR in the factors (5). Finally, we use the "restricted model" where we have imposed the three economic restrictions (i.e., global asset pricing, carry-trade fundamentals and a shrinkage parameter of 0.5) on the prices of risk and combined the resulting restricted estimates with the risk neutral parameters as in (23).

The long-run projections of short-term interest rates implied by the unrestricted model are essentially flat as the largest eigenvalue of the estimated $\mathbf{\Phi}$ in the physical dynamics in equation (5) is well below unity (0.8383). This implies that investors were anticipating much of the drop in interest rates that occurred during the sample. For example, the unrestricted model implies that investor's ten-year ahead expectations of the U.S. short-term rate, as of July 1981 (when the current one-year rate stood over 15 per cent), were 6.00 per cent. Similar patterns can be found for the rest of the countries.

Once we impose our economic restrictions on the prices of risk, forecasts from the restricted model move closer to those implied by risk neutrality. Long-run expectations of short-term rates become more volatile and investors anticipate much less of the decline in interest rates. Using the restricted model, the ten-year ahead expectation of the U.S. short-term rate, as of July 1981, is almost 10 per cent. The projected short-rates are more persistent as the largest eigenvalue of the estimated $\mathbf{\Phi}$ in the restricted model is now very close to one (0.9968). Thus, by imposing restrictions on the prices of risk, we are pulling the largest eigenvalue of $\mathbf{\Phi}$ closer to that of $\mathbf{\Phi}_{11}^{Q}$, which is equal to one. The results are consistent with those for the U.S. term structure in Duffee (2010) who finds that by constraining implied Sharpe ratios, investors tend to anticipate less of the fall in interest rates than in unrestricted models.

We note the large differences across models for the forecasts of the one-year yield during the current financial crisis. The forecasts from the unrestricted model remain flat, indicating that the decline in the short term interest rates that occurred during the crisis were viewed as being temporary. In contrast, the forecasts from the restricted model show

[21] Note that this term risk premium contrasts with the one used in our estimation of Sharpe ratios, which was the expected excess return to holding an n-period bond for one year (13).

a decline in the long run forecasts of the short rate, indicating a very different view of future monetary policy. Of note, the decline in the interest rate forecasts for the Euro area are much smaller than those for the U.S., U.K. or Canada, indicating an overall tighter view of monetary policy.

An important finding in this paper is that the forecasts made using the restricted model are consistent with those from survey data. In Figure 5 we plot the average and 95 per cent confidence intervals of the one-year ahead expectations of the 10 year par bond yield from Consensus Economics Inc. for our four countries along with the same yield implied by our restricted model.[22] The forecasts produced by the restricted model are almost always inside the confidence intervals and indeed, very close to the average of the survey forecasts. We note again that the model appears to capture the quick decline in the expected yields that occurred during the current crisis.

We can evaluate the model's forecasts more formally by a standard test of unbiased expectations. We project the average value of the survey expectations on the model's implied forecast and the sixteen variables in the model (the eight bond market factors and the eight macroeconomic variables that have been orthogonalized with respect to the forecast). Table 7 presents the results for the unrestricted and restricted models. The unrestricted model (Panel A) appears to fit the survey data well, with an estimated coefficient close to 1.00, though the formal test rejects unbiasedness. The sixteen orthogonalized bond and macroeconomic variables are jointly significant in the regression. More important is the economic significance of the model forecast and the additional variables. To show this, we construct variance ratios. The "VR-model" ratio is the variance of the model's forecast divided by the variance of the survey forecast. The ratios range from 0.836 to 0.942 indicating that the unrestricted model does a good job in capturing the variation in the survey data. The ratio labelled "VR-other" is the variance of the linear combination of the sixteen factors from the regression divided by the variance of the survey data. These ratios range from 0.049 to 0.145, indicating that the values of the additional factors have some role to play in explaining the survey data. Clearly, the unrestricted forecasts are not capturing all of the survey data variation.

We can compare this to the forecasts from the restricted model (Panel B). Once again the unbiasedness coefficients are numerically close to 1.00, though statistically different from that value. The VR-model ratios for all four countries examined are above 0.95. The VR-other ratios fall show a large decline. For example, our unrestricted model's forecast explains 83.6 per cent of the variability of the survey forecast for the U.S. 10-year par bond yield. The restricted model explain 95.7 per cent. In addition, the decrease in the VR-other ratio suggest that the information in the (orthogonalized values of the) bond and macroeconomic factors is better incorporated using the restricted version of our model.

The variance ratios thus show that the restrictions make the model fit the survey data better and that the model captures the economically important components of the survey data.

[22]For a more complete description of the Consensus Economics survey data see Jongen, Verschoor and Wolff (2011) for interest rates and Devereux, Smith and Yetman (2012) for exchange rates.

6.1.2 Exchange rates

The model also allows us to make inferences about short and long-run values of expected exchange rates and exchange rate risk premia. We note that the current interest rate differential, $y_{j,t}^{(n)} - y_{\$,t}^{(n)}$, can be decomposed into the (average) expected rate of depreciation over the next n years, $\frac{1}{n}E_t(s_{j,t+n} - s_{j,t})$, and a risk premium term, $fxp_{j,t}^{(n)}$, usually called the foreign exchange premium (see Fama, 1984):

$$y_{\$,t}^{(n)} - y_{j,t}^{(n)} = \frac{1}{n}E_t(s_{j,t+n} - s_{j,t}) + fxp_{j,t}^{(n)}. \tag{25}$$

The foreign exchange risk premium is (the negative of) the expected excess rate of return to a domestic investor for holding an n-year foreign zero-coupon bond. As noted above, unspanned variables do not affect the interest rates but have off-setting effect on the term structure risk premia and expected interest rates. The decomposition in (25) shows that any hidden factor in the yield curves might also appear in expected exchange rates changes and foreign exchange risk premia. Once again, we will compare the model's forecasts to those from survey data to ensure that the decomposition is plausible.

Figure 6 plots the current path of the bilateral exchange rate of the U.S. dollar against the British pound, the Euro and the Canadian dollar, as well as the ten-year ahead projection of the exchange rates generated by the three different models (risk-neutral, unrestricted VAR and our restricted affine term structure model). A clear message emerges from these figures: long-run projections of exchange rates implied by an unrestricted VAR appear unrealistic. For example, in January 2009, when the Euro - U.S. dollar exchange rate stood at approximately 1.28, an investor using the unrestricted model would have forecast a level of 2.03 for January 2019. Similar implications can be found for the other two currencies.

When we impose the restrictions on the prices of risk, the forecasts become closer to those from the risk-neutral model. In addition, they become closer to the current value of the exchange rate. This suggests that our model would produce forecasts of future exchange rates that would be similar to those produced by the random walk model.[23] In addition, as with the interest rate forecasts, we find that our restricted model provides exchange rate forecasts that are consistent with survey data. Figure 7 plots the expected level of the exchange rate one year from now implied by both the restricted and unrestricted models along with the average forecast from Consensus Economics Inc.[24] There is a close relationship between the forecast produced by the restricted model and the survey data. In contrast, the unrestricted model is not able to match the data, at time producing large deviations from the survey data. The differences are particularly large during the recent financial crisis. For example, the unrestricted model produced one-year ahead forecasts of the U.S. dollar against the Euro that ranged as high as 1.80. The restricted model forecasts are much closer to the survey data values of approximately 1.20.

The formal tests of unbiasedness show an even larger effects of the restrictions for the exchange rate forecasts than they did for the interest rate forecasts. The unbiasedness

[23] It has been difficult to produce better out-of-sample forecasts than those from the random walk model (e.g., Meese and Rogoff (1983), Cheung, Chinn and Pascual (2005), and Engel, Mark and West (2007)).

[24] To the best of our knowldege, there is no measure of the dispersions of these forecasts available.

tests for the forecasts from the unrestricted model (Panel A) have slope coefficients that are relatively low, ranging from 0.586 to 0.878. The VR-model ratios indicate that the model's forecasts miss some of the action in the survey data as the ratios range from 0.650 to 0.858. In addition, the orthogonalized values of the bond factors and macroeconomic variables have an economically large effect with VR-other ratios from 0.131 to 0.263.

When the restrictions are imposed (Panel B), the model appears to fit the survey forecasts much better. The coefficients on the model's forecasts rise to 0.735 to 0.860. The VR-model ratio rises dramatically to levels above 91 per cent while the VR-other ratios decline to values less than 7 per cent. Thus, the restrictions help us to obtain forecasts of exchange rates that are consistent with the survey data.

6.1.3 Discussion

We have used the Consensus Economics survey data as a benchmark to show that our restricted model can provide reasonable decompositions of forward curves. Our paper is thus complementary to the growing literature that uses survey data in the estimation of term structure models. Kim and Orphanides (2005) note that traditional term structure models have difficulty matching the persistence of the interest rate process given the short data samples available (e.g., Ball and Torous (1996), Duffee and Stanton (2008), Bauer, Rudebusch and Wu (2011)). Incorporating interest rate survey data in the model's estimation helps to capture the persistence of the yields under the physical measure. Chun (2011) uses survey data on inflation, real growth and interest rates to construct a forward-looking Taylor rule inside an affine term structure model. Chernov and Mueller (2011) use survey based forecasts of inflation to help identify an unspanned variable in the U.S. term structure. Their hidden factor contains information beyond that in the Cochrane and Piazzesi (2005) and Ludvigson and Ng (2009) factors. However, they are not able to relate their factor to interpretable macroeconomic variables. In contrast to these papers, we do not use the survey data in the estimation of the model.

A number of papers use survey data as measures of subjective expectations that are distinct from model based ones. Interest rate survey data have been used by Froot (1989) to evaluate the expectations hypothesis. Bacchetta, Mertens and van Wincoop (2009) examine whether expectational errors that are present in survey data are related to OLS based estimates of time-varying risk premia. Piazzesi and Schneider (2011) use forecasts of interest rates to calculate subjective forecasts of interest rates and compare them to model driven ones. Frankel and Froot (1989) and Chinn and Frankel (2000) use foreign exchange survey data to help explain deviations from uncovered interest parity. Gourinchas and Tornell (2004) use survey data on interest rates to show how distorted beliefs about interest rate shocks can help explain the exchange rate risk premium.

To the best of our knowledge, we are the first paper to show that an international affine term structure model with economic restrictions can produce in-sample forecasts of interest rates and exchange rates that match those from survey data. Financial market participants appear to form expectations incorporating the restrictions of global asset pricing, carry-trade fundamentals and maximal Sharpe ratios that are present in our model. This also suggests that the forecasters surveyed by Consensus Economics are forming their expectations in a rational manner.

We do not make any claim about the out-of-sample forecasting ability of our model. The results presented here are in-sample from a model with a large number of parameters. The model is very unlikely to beat a more parsimonious candidate (e.g., the random walk model) in an out-of-sample evaluation of forecast ability. While our forecasts match those from survey data, the literature shows that survey based forecasts have mixed results in out-of-sample forecasts evaluations. Jongen, Verschoor and Wolff (2011) have shown that the Consensus Economics survey forecasts of interest rates can beat the random walk model in an out-of-sample evaluation. Devereux, Smith and Yetman (2012) show that the Consensus Economics forecasts of exchange rates and inflation rates produce out-of-sample forecasts of the real exchange rate that are unbiased when averaged over a number of countries. Ang, Bekaert and Wei (2007) show that survey based forecasts of inflation beat those produced by a variety of statistical models. However, there is other evidence that survey data may not yield unbiased forecasts (Pesaran and Weale (2006)).

6.2 Risk premia

6.2.1 Interest rates

Given that the restricted model is able to match the expected interest rates from the survey data, the decomposition in (24) indicates that we will have more accurate estimates of the international term premia than those provided by unrestricted models. Figure 8 plots the forward term premia implied by our restricted model for all four countries. Panel A shows the forward term premia on an "in-2-for-1" loan (i.e., a one-year loan initiated in two years), $f_{j,t}^{(3)}$, while Panel B plots the forward term premia on an "in-9-for-1" loans (i.e., a one-year loan initiated in nine years), $f_{j,t}^{(10)}$. Both panels also display NBER U.S. recession dates. For a given maturity, term premia movements across countries follow each other tightly with a correlation close to one. This indicates that the imposition of global asset pricing on the one-year holding period returns carries through to longer maturities. However, for a given country, the correlation of term premia across maturities is only 0.8. For example, the "in-2-for-1" forward premium in the U.S. during the 2008-2009 recession was as high as it was in the early 1980s. On the other hand, the "in-9-for-1" forward premium during the same recession was only half of that observed during the early 1980s. This result can be interpreted as evidence in favour of more than one global factor in term premia.

Forward term premia tend to drift downwards from the early 1980s until the early 2000s for all countries and maturities. The estimated term premia are countercyclical: they are close to zero or even negative just before the major crises of 1980 and 2008 and increase rapidly during recessions, including the recent financial crisis.

Our results using the restricted model match the long downward trend in term premia found in JPS for the U.S. term premia and in Wright (2011) for the ten countries in his study. There are some differences, however. For example, the cross section of international term premia are much more correlated in our study than in Wright's due to the use of global asset pricing. Wright (2011) also finds that the term premium component of forward rates remain flat (or even fell) during 2009. Using our model, term premia increase.[25]

[25] Yet, it is important to notice that Wright (2011) also computes estimates of the term premia using

Finally, it is worth noting that our results reveal an international dimension to the "conundrum". The conundrum was first raised as a puzzle by Alan Greenspan who noted that long-term U.S. interest rates remained low despite of the tightening monetary policy during the 2004-2005 period. Kim and Wright (2005), Rudebusch, Swanson and Wu (2006), Backus and Wright (2007), and Cochrane and Piazzesi (2008) have examined the conundrum from the U.S. perspective. The conclusion from this line of research is that term premia declined at the same time that the Federal Reserve was raising the policy rate during the mid 2004 to mid 2005 period.

We can examine the international aspects of the conundrum by decomposing the country j's n-year bond yield into an expectation and a term premia component, $tp_{j,t}^{(n)}$, across each of the j countries:

$$y_{j,t}^{(n)} = \frac{1}{n}\sum_{h=1}^{n} E_t y_{j,t+h-1}^{(1)} + tp_{j,t}^{(n)}. \tag{26}$$

We can use this decomposition to decompose an observed ten-year yield, say, into the expectation of the average short-rate over the next 10 years (the first term in (26)) and the associated term structure risk premium (the second term).

Table 8 provides the decompositions for the four countries over the May 2004 to July 2005 period. During this time the short-run (one year) interest rate in the U.S. increased by 222 basis points. We note that while the short-rate in Canada also increased (75 basis points) the short rates in the U.K. and the Euro area decreased slightly. At the same time, there was a decrease in the ten-year yield in the U.S. by 41 basis points that was accompanied by even larger decreases in the other countries (e.g. a fall of 113 basis points for German bonds). The model indicates that market participants had differing views on expected short-term interest rates in the four countries.

However, the model is consistent in its calculations of a decline in international term premia, even though the May 2004 values of the risk premia are quite low (the largest estimated value was 90 basis points for U.S. bonds). The model indicates declines in term premia ranging from 50 basis points for German bonds to 83 basis points for Canadian bonds over the subsequent year. Clearly the conundrum should be viewed as being part of a "conundra" (Detken (2006)) as global risk premia declined.

Of note, the estimated residuals from the model are quite small, ranging from -10.59 to 19.62 basis points. These are much smaller than the estimated residuals in Rudebusch, Swanson and Wu (2006) for example, suggesting that the period under study is not anomalous from the restricted model's perspective.

6.2.2 Exchange rates

Figure 9 shows the foreign exchange risk premia (25) for the three bilateral exchange rates over a 10 year forecast horizon. The estimated foreign exchange risk premia are quite volatile. The premia are counter-cyclical to the U.S. economy, increasing during recessions, especially for the Euro and Canadian dollar exchange rates. We note that

survey data and that these estimates spike briefly during 2009 (at least for the four countries that we study in this paper). In addition, Wright (2011) focuses on the term premia on "in-5-for-5" loans (five-year loans initiated in five years). Our results remain qualitatively the same when we compute such estimates.

during the latest crisis, the risk premia on the Euro and Canadian dollar initially fell while that on the pound rose. This may be the result of the different monetary policy stances taken by the central banks of the four countries. There was an initial flight to quality into U.S. dollars which caused an appreciation of the currency against the others at the start of the crisis. Subsequently, the U.S. Federal Reserve and the Bank of England both undertook quantitative easing policies which may have resulted in a flight to quality into other currencies.

We can use our model to decompose the foreign exchange risk premia, $fxp_{j,t}^{(n)}$, into a pure currency risk premia component and a term that reflects compensation for interest rate risk. Substituting the decomposition of term premia in (26) into equation (25) and rearranging, we find that

$$fxp_{j,t}^{(n)} = \frac{1}{n}\sum_{h=1}^{n} E_t fxp_{j,t+h-1}^{(1)} - (tp_{\$,t}^{(n)} - tp_{j,t}^{(n)}). \tag{27}$$

The n-year foreign exchange risk premia is equal to the average of the path of the one-year foreign exchange premia over the next n years and the difference between the term premia in each one of the countries. When there is no interest risk, term premia are equal to zero. Thus, the first term in (27) reflects a pure currency risk component.[26]

Figure 9 also shows the decompositions for the three exchange rates. The interest rate risk component of the foreign exchange risk premia are quite small. Most of the action is the result of the currency component. This may be due to our combined assumptions of global asset pricing and carry-trade fundamentals. Global asset pricing results in interest rates that are determined by global factors. The carry-trade fundamentals assumption uses the difference in one-year interest rates in foreign exchange risk premia. As a result, the term premia components of exchange rate risk will be determined by the difference in loadings between the U.S. and foreign country term premia on the global factor. If the loadings are of similar magnitude, the net effect on exchange rate risk premia will be small.

6.3 Unspanned macroeconomic risks

To gauge the importance of unspanned macroeconomic risks, we calculate variance decompositions of forward term structure and foreign exchange risk premia and the associated expected interest rates and exchange rates. We follow JPS and focus on the conditional variances to account for the near unit root behavior present in the physical dynamics. This allows us to remove the trend like component from our series and isolate how news in bond, macro and foreign exchange factors contribute to the risk premia and expectations variability.

We start by exploring the role of unspanned risks in expected interest rates and forward term premia. We use a Cholesky decomposition of $\mathbf{\Sigma}$ subject to two different orderings of the factors $\mathbf{x}_t$ to identify the proportion of the conditional volatility of both expected interest rates and forward term premia from (24) explained by the macroeconomic factors,

[26] A similar expression has been derived by Bekaert and Hodrick (2001) for the case of constant risk premia, and by Sarno, Schneider and Wagner (2011).

$\mathbf{m}_t$. In the first ordering, $(\mathbf{f}_t, \Delta\mathbf{s}_t, \mathbf{m}_t)$, we compute the proportion of the conditional volatility that is explained by the components of the macroeconomic factors that are orthogonal to the yield curve factors and exchange rates (i.e., the unspanned component of the macroeconomic factors). We also compute the proportion of the variability that is explained the component of inflation that is orthogonal to growth, to yield curve factors and to rates of depreciation (i.e., the unspanned components of the inflation variables). We thus assume the following ordering within the macroeconomic variables: $\mathbf{m}_t' = (\mathbf{g}_t', \boldsymbol{\pi}_t')'$, where we collect the growth rates ($\mathbf{g}_t$) and the inflation rates ($\boldsymbol{\pi}_t$) for the four countries.[27]

In the second ordering, we calculate the proportion of the variability that is due to both the spanned and unspanned components of the macroeconomic factors $\mathbf{m}_t$ (i.e., the total effect of the macroeconomic factors) by shocking the macroeconomic variables in $(\mathbf{m}_t, \mathbf{f}_t, \Delta\mathbf{s}_t)$.

Panel A in Figure 10 shows the one-year ahead variance decompositions of the expected interest rates (left hand column) and the forward term premia (right hand column) for maturities ranging from 2 to 10 years and for the four countries using our restricted model. The total effect of macroeconomic variables is very large on the expected interest rates, with over 40 per cent of the variation explained for all four countries across all horizons. Notably absent, however, is any effect of unspanned macroeconomic variables (or the unspanned portion of foreign exchange rates). Thus, expected future interest rates are explained in large part by that portion of the variation in macroeconomic variables that is related to the bond market factors. This is not too surprising given the large literature that uses macroeconomic variables such as growth and inflation in Taylor-type rules for monetary policy.

The striking finding is the very large impact of macroeconomic variables on international term premia. For example, over 80 per cent of the one-year ahead variance of two-year forward term premia, $ftp_{j,t}^{(2)}$, is explained by the total component of the macroeconomic factors for all four countries. The effect of the macroeconomic variables declines with maturity. Around 30 per cent of the one-year ahead variability of ten-year forward term premia, $ftp_{j,t}^{(10)}$, is explained by the total component of the macroeconomic factors. This result of declining power over longer horizons mirrors that of the domestic U.S. term structure as found in JPS.

The effect of macroeconomic variables remains large even after they are orthogonalized with respect to the bond market factors and the exchange rates. Unspanned macroeconomic factors explain over 50 per cent of the variability of the one-year ahead variance of two-year forward premia and approximately 30 per cent of long-dated forward term premia. This shows the importance of including unspanned variables in the model. We note that we find a similar pattern across the four countries due to the assumption of global asset pricing that was also evident in Figure 5.

The proportion of variance explained by the unspanned inflation has a hump-shaped pattern in the one-year ahead decompositions, peaking at approximately year three. For example, almost 45 per cent of the variability of three-year forward term premiums,

[27] We note that unspanned macroeconomic risks in domestic term structure models are orthogonalized to the bond market factors only. In our international application, we chose to orthogonalize the macroeconomic factors with respect to exchange rates as well. We do so as it will provide a more conservative estimate of the effects of macroeconomic risks on asset prices.

$ftp_{j,t}^{(3)}$, is explained by the unspanned component of inflation. Also note that the lines for unspanned macro and unspanned inflation converge as we focus on longer-dated forward loans. As such, the effect of unspanned macro risks on the term premia of forward loans far in the future reflects mainly inflation risk. The effect of unspanned foreign exchange risk is quite small, accounting for less than 10 per cent of the variation in forward term premia.

The same Cholesky decompositions can be used to measure the effects of unspanned and spanned macroeconomic risks on the conditional variance of the expected exchange rates and foreign exchange risk premia from (25). Panel B in Figure 10 shows the decompositions for the expected exchange rates (left hand column) and the foreign exchange risk premia (right hand column) for holding periods out to 10 years. A surprising result is the large effect of macroeconomic shocks on the expected rate of depreciation shown in the left column. Both the unspanned and total shocks account for a large part of the expected rates of depreciation. For example, the total effect of both macroeconomic variables accounts for over 90 per cent of the one year variation in the U.S. dollar - British pound exchange rate. A large portion of this comes from the unspanned portion of the macroeconomic variables. Indeed, the unspanned components of the variables are responsible for large amounts of variation in expected one-year change of the U.S. dollar/Euro (over 80 per cent) and U.S. dollar/Canada dollar (over 60 per cent) exchange rates as well. As the forecast interval lengthens, the total effect of the macroeconomic variables declines, with ratios reaching below 20 per cent for the U.S. dollar/Euro and U.S. dollar/Canada dollar exchange rates.

We note that these results appear to contrast with the usual findings of the "disconnect puzzle" that originated with Meese and Rogoff (1983). A subsequent large literature has shown that macroeconomic variables have not been found to be important drivers of exchange rates, except perhaps at long horizons (e.g., Mark (1995), Cheung, Chinn and Pascual (2005), and Sarno (2005)). We note several differences between previous approaches and the current model. First, this is the first paper to examine the role of the unspanned portions of the macroeconomic variables for exchange rates. Second, there is a lot of structure in our model that is absent in other work, particularly the restrictions on the prices of risk. Third, the forecasts of the exchange rates from the model match those from the Consensus Economics survey indicating a certain level of reasonableness. Previous efforts to relate exchange rates to macroeconomic variables have not included these ingredients. It may well be that these are individually or jointly crucial to understanding the disconnect puzzle.

We view our results as complementing the analysis of Engle and West (2005) and Sarno and Solji (2009). In these papers the exchange rate is related to expectations of future relative fundamentals, which are discounted using a large, but constant, discount rate. As a result, the exchange rate appears to have near random walk behavior and thus not related to macroeconomic variables. In contrast, our paper links exchange rates to bond market and macroeconomic fundamentals within a stochastic discount factor framework with time varying prices of risks. As noted above, the restricted model's exchange rate forecasts also present a similar near random walk behavior. However, the model shows a large impact of unspanned macroeconomic variables on the portion of exchange rate changes that is anticipated by the model. Given the large role of unspanned exchange rate risk, the model will not explain a large part of the realized variation in exchange rates.

Macroeconomic variables (both the total and unspanned components) also explain a large amount of the one-year ahead forecast variance of the foreign exchange risk premia for the Euro and the Canadian dollar. For example, almost 100 per cent of the variance of one-year foreign exchange risk premium for the Euro, $fxp_{j,t}^{(1)}$, is explained by the total contribution of the macroeconomic factors. The contribution for the Canadian dollar exchange rate premium is close to 60 per cent. When we orthogonalize the macroeconomic variables with respect to the information contained in the yield curve, the unspanned components explain close to 50 per cent of the variability of $fxp_{j,t}^{(1)}$ for the Euro and over 25 per cent for the Canadian dollar. However, the proportion of variability of the risk premia explained by macroeconomic variables declines with maturity. It is also interesting to note that unspanned inflation also seem to be an important driver of the Euro while accounting for less of the variation in the other two currencies. The macroeconomic variables seem to explain little of the variability of the British pound premia with only 25 per cent of the forecast variance of the foreign exchange premia (for any given maturity) seem to be explained by the total effect of macroeconomic factors.

7 Final remarks

This paper builds an international dynamic term structure model to show the important role of unspanned risks in explaining the links between global macroeconomic fundamentals and the cross section of international interest rates and exchange rates. We find that it is important to impose economic restrictions (including global asset pricing, carry trade fundamentals and maximal Sharpe ratios) on the prices of risk to obtain plausible decompositions of forward curves into an expectation and risk premia component. This, in turn, enables us to identify the (offsetting) effects of the unspanned variables in the two components. We verify that our model produces reasonable decompositions by showing that the restricted model's forecasts of interest rates and exchange rates match those from survey data.

Our results reveal that it is the global component of the (unspanned) macroeconomic variables that drives term structure risk premia. In addition, and new to the foreign exchange literature, we find that the unspanned component of macroeconomic variables have a relatively large effect on foreign exchange risk premia and expected exchange rate changes. We view our results as suggestive for further research on the links between macroeconomic variables and exchange rates using modern asset pricing methods.

Appendix

A Inter-battery factor analysis

Let $\mathbf{y}_{jt}$ be the vector of $N \times 1$ vector of observable variables for each block (i.e. bond yields in each country) and assume the following joint model for $\mathbf{y}_{jt}$ and $j = 1, \ldots, J$, where J is the number of blocks (i.e. countries):

$$\begin{pmatrix} \mathbf{y}_{1t} \\ \vdots \\ \mathbf{y}_{Jt} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \vdots \\ \boldsymbol{\mu}_J \end{pmatrix} + \begin{pmatrix} \mathbf{B}_1 \\ \vdots \\ \mathbf{B}_J \end{pmatrix} \mathbf{f}_t + \begin{pmatrix} \boldsymbol{\varepsilon}_{1t} \\ \vdots \\ \boldsymbol{\varepsilon}_{Jt} \end{pmatrix} \tag{A.1}$$

$$\begin{pmatrix} \mathbf{f}_t \\ \boldsymbol{\varepsilon}_{1t} \\ \vdots \\ \boldsymbol{\varepsilon}_{Jt} \end{pmatrix} \sim i.i.d.\ N \left[\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{I}_K & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi}_{11} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \boldsymbol{\Psi}_{JJ} \end{pmatrix} \right] \tag{A.2}$$

where $\mathbf{f}_t$ is a $K \times 1$ vector of unobserved common factors that are orthogonal to each other, such that $E(\mathbf{f}_t\mathbf{f}_t') = \mathbf{I}_K$ with $\mathbf{I}_K$ a K-dimensional identity matrix; $\mathbf{B}_j$ is a $N \times K$ matrix of constant loadings on the common factors; and $\boldsymbol{\varepsilon}_{jt}$ is a $N \times 1$ vector of idiosyncratic noises with zero, which conditionally orthogonal to $\mathbf{f}_t$ and to any other $\boldsymbol{\varepsilon}_{lt}$ for $l \neq j$. The main difference with respect to a traditional factor model is that we do not assume that the covariance matrix of the idiosyncratic noise for each country ($E(\boldsymbol{\varepsilon}_{jt}\boldsymbol{\varepsilon}_{jt}') = \boldsymbol{\Psi}_{jj}$) is diagonal. Such an assumption implies the potential presence of factors being common to one block only (i.e., a country-specific factor).

Stacking across blocks, let $\mathbf{y}_t = (\mathbf{y}_{1t}', \ldots, \mathbf{y}_{Jt}')'$, $\boldsymbol{\mu} = (\boldsymbol{\mu}_1', \ldots, \boldsymbol{\mu}_J')'$, $\mathbf{B} = (\mathbf{B}_1', \ldots, \mathbf{B}_J')'$, and $\boldsymbol{\varepsilon}_t = (\boldsymbol{\varepsilon}_{1t}', \ldots, \boldsymbol{\varepsilon}_{Jt}')'$ so that the model in equations (A.1) and (A.2) can be expressed in compact form as

$$\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{B}\mathbf{f}_t + \boldsymbol{\varepsilon}_t \tag{A.3}$$

$$\begin{pmatrix} \mathbf{f}_t \\ \boldsymbol{\varepsilon}_t \end{pmatrix} \sim i.i.d.\ N \left[\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Psi} \end{pmatrix} \right] \tag{A.4}$$

where $\boldsymbol{\Psi}$ is a $NJ \times NJ$ block diagonal matrix, with relevant block given by $\boldsymbol{\Psi}_{jj}$. These assumptions imply that the distribution of $\mathbf{y}_t$ is multivariate *i.i.d.* normal with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma} = E(\mathbf{y}_t\mathbf{y}_t') = \mathbf{B}\mathbf{B}' + \boldsymbol{\Psi}$.

Pérignon, Smith and Villa (2007) suggest estimating this common factor model by maximum likelihood. They undertake a numerical maximization of the log-likelihood of a sample of size T of the observable variables:

$$\ln L_T(\boldsymbol{\mu}, \mathbf{B}, \boldsymbol{\Psi}) = \sum_{t=1}^{T} l_t$$

with

$$l_t = -\frac{NJ}{2}\log 2\pi - \frac{1}{2}\ln|\mathbf{B}\mathbf{B}' + \boldsymbol{\Psi}| - \frac{1}{2}(\mathbf{y}_t - \boldsymbol{\mu})'(\mathbf{B}\mathbf{B}' + \boldsymbol{\Psi})^{-1}(\mathbf{y}_t - \boldsymbol{\mu})$$

Given the dimension of our problem (four countries and ten maturities per country), such numerical optimization is infeasible. For this reason, we adapt the EM algorithm approach of Sentana (2000), used with conditionally heteroskedastic exact factor models, to our inter-battery factor analysis framework.[28]

To this end, notice that the model given by (A.3) and (A.4) has a state space representation where $\mathbf{f}_t$ can be regarded as the state. Under such a representation, (A.3) is the measurement equation given that it describes the relation between the observed variables $\mathbf{y}_t$ and the unobserved factor $\mathbf{f}_t$. The transition equation, on the other hand, has a degenerate nature as it can be represented as $\mathbf{f}_t = \mathbf{0} \cdot \mathbf{f}_{t-1} + \mathbf{f}_t$. We can thus apply the Kalman filter in order to obtain the best (in the conditional mean square error sense) estimate of the factor $\mathbf{f}_{t|t} = E(\mathbf{f}_t | \mathbf{y}_t)$ and the corresponding mean squared errors $\mathbf{\Omega}_{t|t} = V(\mathbf{f}_t | \mathbf{y}_t)$:

$$\mathbf{f}_{t|t} = \mathbf{B}'\mathbf{\Sigma}^{-1}(\mathbf{y}_t - \boldsymbol{\mu}) \tag{A.5}$$

$$\mathbf{\Omega}_{t|t} = \mathbf{I}_k - \mathbf{B}'\mathbf{\Sigma}^{-1}\mathbf{B} \tag{A.6}$$

Further notice that, given that the transition equation is degenerate, smoothing is unnecesary so that $\mathbf{f}_{t|t} = E(\mathbf{f}_t | \mathbf{Y}_T)$, and $\mathbf{\Omega}_{t|t} = V(\mathbf{f}_t | \mathbf{Y}_T)$ where $\mathbf{Y}_T = \{\mathbf{y}_T, \mathbf{y}_{T-1}, \ldots\}$.

The EM algorithm is based on the following identity:

$$\begin{aligned} l(\mathbf{y}_t, \mathbf{f}_t; \boldsymbol{\phi}) &\equiv l(\mathbf{y}_t | \mathbf{f}_t; \boldsymbol{\phi}) + l(\mathbf{f}_t; \boldsymbol{\phi}) \\ l(\mathbf{y}_t, \mathbf{f}_t; \boldsymbol{\phi}) &\equiv l(\mathbf{f}_t | \mathbf{y}_t; \boldsymbol{\phi}) + l(\mathbf{y}_t; \boldsymbol{\phi}) \end{aligned} \tag{A.7}$$

where $l(\mathbf{y}_t, \mathbf{f}_t; \boldsymbol{\phi})$ is the joint log-density function of $\mathbf{y}_t$ and $\mathbf{f}_t$; $l(\mathbf{y}_t | \mathbf{f}_t; \boldsymbol{\phi})$ is the conditional log-density of $\mathbf{y}_t$ given $\mathbf{f}_t$; $l(\mathbf{f}_t; \boldsymbol{\phi})$ the marginal log-density of $\mathbf{f}_t$; $l(\mathbf{f}_t | \mathbf{y}_t; \boldsymbol{\phi})$ the conditional log-density of $\mathbf{f}_t$ given $\mathbf{y}_t$; and $l(\mathbf{y}_t; \boldsymbol{\phi})$ the marginal log-density of $\mathbf{y}_t$, given the parameters vector of parameters of the model, $\boldsymbol{\phi}$. In particular, the EM algorithm exploits the Kullback inequality which states that any increase in $E\left[\sum_t l(\mathbf{y}_t, \mathbf{f}_t | \mathbf{Y}_T; \boldsymbol{\phi})\right]$ must represent an increase in the log-likelihood of the sample $\sum_t l(\mathbf{y}_t | \mathbf{Y}_T; \boldsymbol{\phi})$. The essential steps of this algorithm are the E(stimation)-Step and the M(aximisation)-Step which are carried out at each iteration. So at the n-th iteration we have:

E-Step: Given the current value $\boldsymbol{\phi}^{(n)}$ of the parameter vector and the observed data $\mathbf{Y}_T$, calculate estimates for $\mathbf{f}_t$ as $E(\mathbf{f}_t | \mathbf{Y}_T, \phi^{(n)})$. This estimate coincides with $\mathbf{f}_{t|t}$ evaluated at $\phi^{(n)}$ and could, in principle be obtained from equation (A.5). Yet, as noted by Sentana (2000), a much more efficient algorithm can be obtained by exploiting the identity in (A.7). From this identity, and after grouping terms we can write:

$$\begin{aligned} l(\mathbf{y}_t; \boldsymbol{\phi}) = & -\frac{NJ}{2} \log 2\pi - \frac{1}{2} \log\left(|\mathbf{\Psi}| \cdot \left|\mathbf{\Omega}_{t|t}^{-1}\right| \right) \\ & -\frac{1}{2}\left[(\mathbf{y}_t - \boldsymbol{\mu})'\mathbf{\Psi}^{-1}(\mathbf{y}_t - \boldsymbol{\mu}) - \mathbf{f}_{t|t}'\mathbf{\Omega}_{t|t}^{-1}\mathbf{f}_{t|t} \right] \\ & -\frac{1}{2}\mathbf{f}_t'\left(\mathbf{I}_k + \mathbf{B}'\mathbf{\Psi}^{-1}\mathbf{B} - \mathbf{\Omega}_{t|t}^{-1} \right)\mathbf{f}_t - \frac{1}{2}\mathbf{f}_t'\left[\mathbf{\Omega}_{t|t}^{-1}\mathbf{f}_{t|t} - \mathbf{B}'\mathbf{\Psi}^{-1}(\mathbf{y}_t - \boldsymbol{\mu}) \right] \end{aligned} \tag{A.8}$$

[28] Nevertheless, our problem is much more simpler given that we are supposing an *i.i.d.* sample.

provided that the necessary inverses exits. But since marginal log-density of $\mathbf{y}_t$ cannot depend on $\mathbf{f}_t$, the last two terms must be indentically 0 for any value of $\mathbf{f}_t$, it follows that

$$\mathbf{\Omega}_{t|t} = (\mathbf{I}_k + \mathbf{B}'\mathbf{\Psi}^{-1}\mathbf{B})^{-1}$$
$$\mathbf{f}_{t|t} = \mathbf{\Omega}_{t|t}^{-1}\mathbf{B}'\mathbf{\Psi}^{-1}(\mathbf{y}_t - \boldsymbol{\mu})$$

As noted by Sentana (2002), using these expressions for $\mathbf{f}_{t|t}$ and $\mathbf{\Omega}_{t|t}$ evaluated at $\boldsymbol{\phi}^{(n)}$ (instead of those in equations A.5 and A.6) is computationally advantageous as we replace the inversion of $\mathbf{\Sigma}$ by the inversion of $\mathbf{\Psi}$ (which is block-diagonal).

M-Step: Using the estimated values $\mathbf{f}_{t|t}^{(n)}$ and $\mathbf{\Omega}_{t|t}^{(n)}$, in this step we maximize the expected value of $\sum_t [l(\mathbf{y}_t|\mathbf{f}_t;\boldsymbol{\phi}) + l(\mathbf{f}_t;\boldsymbol{\phi})]$ conditional on $\mathbf{Y}_T$ and the current parameter estimates $\boldsymbol{\phi}^{(n)}$ to determine $\boldsymbol{\phi}^{(n+1)}$. In particular, the objective function at the M-Step of the n-th iteration is:

$$-\frac{TNJ}{2}\log 2\pi - \frac{T}{2}\log|\mathbf{\Psi}| - \frac{1}{2}\sum_{t=1}^{T} tr\left\{\mathbf{\Psi}^{-1}\left[(\mathbf{y}_t - \boldsymbol{\mu} - \mathbf{B}\mathbf{f}_{t|t}^{(n)})(\mathbf{y}_t - \boldsymbol{\mu} - \mathbf{B}\mathbf{f}_{t|t}^{(n)})' + \mathbf{B}\mathbf{\Omega}_{t|t}^{(n)}\mathbf{B}'\right]\right\}$$
$$-\frac{TK}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T} tr\left[\mathbf{f}_{t|t}^{(n)}\mathbf{f}_{t|t}^{(n)\prime} + \mathbf{\Omega}_{t|t}^{(n)}\right]$$

and, given our assumptions, this implies that:

$$\widehat{\boldsymbol{\mu}} = \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_t \tag{A.9}$$
$$\widehat{\mathbf{B}} = \mathbf{B}^{(n+1)} = \left[\sum_{t=1}^{T}\mathbf{f}_{t|t}^{(n)}\mathbf{f}_{t|t}^{(n)\prime} + \mathbf{\Omega}_{t|t}^{(n)}\right]^{-1}\left[\sum_{t=1}^{T}\mathbf{f}_{t|t}^{(n)}(\mathbf{y}_t - \boldsymbol{\mu})'\right]$$

and that $\mathbf{\Psi}_{jj}^{(n+1)}$ can be obtained from the relevant blocks of

$$\frac{1}{T}\sum_{t=1}^{T}\left[(\mathbf{y}_t - \boldsymbol{\mu} - \mathbf{B}\mathbf{f}_{t|t}^{(n)})(\mathbf{y}_t - \boldsymbol{\mu} - \mathbf{B}\mathbf{f}_{t|t}^{(n)})' + \mathbf{B}\mathbf{\Omega}_{t|t}^{(n)}\mathbf{B}'\right]$$

Note that, estimates of $\boldsymbol{\mu}$ are independent of the iteration and they coincide with the sample means of $\mathbf{y}_t$. Thus, we can safely apply our inter-battery factor analysis to demeaned data. Furthermore, the expressions obtained in the M-step are very similar to those corresponding to the multivariate regression case and that we would apply to the case in which $\mathbf{f}_t$ were observed. In the M-step, instead, the unobservable factors are replaced by their best (in the conditional mean squared error sense) estimates given the available data.

B Bond Pricing

In this appendix, we show that in a model the state variables, $\mathbf{x}_t$, follow a VAR(1) process:

$$\mathbf{x}_{t+1} = \boldsymbol{\mu} + \mathbf{\Phi}\mathbf{x}_t + \mathbf{v}_{t+1}$$

where $\mathbf{v}_t \sim iid\ N(0, \mathbf{\Sigma})$, and where the stochastic discount factor (SDF) is given by

$$\xi_{t+1} = \exp(-r_t - \frac{1}{2}\boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1}\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1}\mathbf{v}_{t+1})$$

with a short rate and prices of risk given by the following affine functions:

$$\begin{aligned} \boldsymbol{\lambda}_t &= \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1\mathbf{x}_t \\ r_t &= \delta_0 + \boldsymbol{\delta}_1'\mathbf{x}_t \end{aligned}$$

the log bond prices are affine functions of the state variables

$$p_t^{(n)} = A_n + \mathbf{B}_n'\mathbf{x}_t$$

where A_n and $\mathbf{B}_n$ can be computed recursively:

$$\mathbf{B}_{n+1}' = \mathbf{B}_n'\mathbf{\Phi}^Q - \boldsymbol{\delta}_1' \tag{B.1}$$

$$A_{n+1} = A_n + \mathbf{B}_n'\boldsymbol{\mu}^Q + \frac{1}{2}\mathbf{B}_n'\mathbf{\Sigma}\mathbf{B}_n - \delta_0 \tag{B.2}$$

with $A_1 = -\delta_1$, $\mathbf{B}_1 = -\boldsymbol{\delta}_1$ and $\boldsymbol{\mu}^Q = \boldsymbol{\mu} - \boldsymbol{\lambda}_0$ and $\mathbf{\Phi}^Q = \mathbf{\Phi} - \boldsymbol{\lambda}_1$ are the matrices governing the dynamic evolution of the state variables under the risk neutral measure.

Note that the affine pricing relationship is trivially satisfied for one-period bonds $(n = 1)$ given that

$$p_t^{(1)} = -y_t^{(1)} = -r_t = -\delta_0 - \boldsymbol{\delta}_1'\mathbf{x}_t$$

For $n > 1$, we have that the price at time t of a $n + 1$ zero-coupon bond is given by

$$P_t^{(n+1)} = E_t[\xi_{t+1}P_{t+1}^{(n)}]$$

and, thus, we must have

$$\begin{aligned} P_t^{(n+1)} &= E_t\left[\exp\left(-r_t - \frac{1}{2}\boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1}\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1}\mathbf{v}_{t+1} + A_n + \mathbf{B}_n'\mathbf{x}_{t+1}\right)\right] \\ &= E_t\left\{\exp\left[-r_t - \frac{1}{2}\boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1}\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1}\mathbf{v}_{t+1} + A_n + \mathbf{B}_n'(\boldsymbol{\mu} + \mathbf{\Phi}\mathbf{x}_t + \mathbf{v}_{t+1})\right]\right\} \\ &= E_t\left[\exp\left(-r_t - \frac{1}{2}\boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1}\boldsymbol{\lambda}_t + A_n + \mathbf{B}_n'\boldsymbol{\mu} + \mathbf{B}_n'\mathbf{\Phi}\mathbf{x}_t + (\mathbf{B}_n' - \boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1})\mathbf{v}_{t+1}\right)\right] \end{aligned}$$

Note, however, that the last term in the previous equation satisfies that

$$\begin{aligned} E_t\left\{\exp\left[(\mathbf{B}_n' - \boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1})\mathbf{v}_{t+1}\right]\right\} &= \exp\left[\frac{1}{2}(\mathbf{B}_n' - \boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1})\mathbf{\Sigma}(\mathbf{B}_n' - \boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1})\right] \\ &= \exp\left(\frac{1}{2}\boldsymbol{\lambda}_t'\mathbf{\Sigma}^{-1}\boldsymbol{\lambda}_t - \mathbf{B}_n'\boldsymbol{\lambda}_t + \frac{1}{2}\mathbf{B}_n'\mathbf{\Sigma}\mathbf{B}_n\right) \end{aligned}$$

Thus we have that

$$\begin{aligned} A_{n+1} + \mathbf{B}_{n+1}'\mathbf{x}_t &= -\delta_0 + A_n + \mathbf{B}_n'\boldsymbol{\mu} - \mathbf{B}_n'\boldsymbol{\lambda}_0 + \frac{1}{2}\mathbf{B}_n'\mathbf{\Sigma}\mathbf{B}_n - \boldsymbol{\delta}_1'\mathbf{x}_t + \mathbf{B}_n'\mathbf{\Phi}\mathbf{x}_t - \mathbf{B}_n'\boldsymbol{\lambda}_1\mathbf{x}_t \\ &= \left[A_n + \mathbf{B}_n'(\boldsymbol{\mu} - \boldsymbol{\lambda}_0) + \frac{1}{2}\mathbf{B}_n'\mathbf{\Sigma}\mathbf{B}_n - \delta_0\right] + \left[\mathbf{B}_n'(\mathbf{\Phi} - \boldsymbol{\lambda}_1) - \boldsymbol{\delta}_1'\right]\mathbf{x}_t \end{aligned}$$

And matching coefficients we arrive at the pricing equations above.

C Exchange rates and stochastic discount factors

C.1 Uncovered interest parity under the risk-neutral measure

In this section, we show that the fact that uncovered interest parity must hold under the risk-neutral measure is a direct consequence of the pricing equation of a foreign one-period zero-coupon bond by a domestic investor. In particular, note that the assumption of no-arbitrage implies that the price of a foreign one-period zero-coupon bond, $P_{j,t}^{(1)} = e^{-r_{j,t}}$, must satisfy:

$$E_t\left(\xi_{\$,t+1}e^{\Delta s_{j,t+1}} \times 1/P_{j,t}^{(1)}\right) = E_t\left(\xi_{\$,t+1}e^{\Delta s_{j,t+1}}e^{r_{j,t}}\right) = 1$$

and substituting the law of motion for the rate of depreciation (equation 5 in the main text) and the domestic SDF (equation 6 in the main text) into this last equation

$$\begin{aligned} E_t\left\{\exp\left[-r_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1} + \mathbf{e}_{F+M+j}'(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t + \mathbf{v}_{t+1}) + r_{j,t}\right]\right\} &= 1 \\ E_t\left\{\exp\left[(r_{j,t} - r_{\$,t}) - \frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_t + \mathbf{e}_{F+M+j}'(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t) + (\mathbf{e}_{F+M+j}' - \boldsymbol{\lambda}_t'\boldsymbol{\Sigma}^{-1})\mathbf{v}_{t+1}\right]\right\} &= 1 \end{aligned}$$

Note, however, that the last term in the previous equation satisfies that

$$\begin{aligned} E_t\left\{\exp\left[(\mathbf{e}_{F+M+j}' - \boldsymbol{\lambda}_t'\boldsymbol{\Sigma}^{-1})\mathbf{v}_{t+1}\right]\right\} &= \exp\left[\frac{1}{2}(\mathbf{e}_{F+M+j}' - \boldsymbol{\lambda}_t'\boldsymbol{\Sigma}^{-1})\boldsymbol{\Sigma}(\mathbf{e}_{F+M+j}' - \boldsymbol{\lambda}_t'\boldsymbol{\Sigma}^{-1})\right] \\ &= \exp\left(\frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_t - \mathbf{e}_{F+M+j}'\boldsymbol{\lambda}_t + \frac{1}{2}\mathbf{e}_{F+M+j}'\boldsymbol{\Sigma}\mathbf{e}_{F+M+j}\right) \end{aligned}$$

which implies that

$$(r_{j,t} - r_{\$,t}) + \mathbf{e}_{F+M+j}'(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t) - \mathbf{e}_{F+M+j}'\boldsymbol{\lambda}_t + \frac{1}{2}\mathbf{e}_{F+M+j}'\boldsymbol{\Sigma}\mathbf{e}_{F+M+j} = 0.$$

Substituting the expressions for the short-term interest rates and the prices of risk into this equation delivers the following conditions for all $j = 1, ..., J$:

$$\mathbf{e}_j'\boldsymbol{\mu}_3^Q = -\frac{1}{2}\mathbf{e}_j'\boldsymbol{\Sigma}_{33}\mathbf{e}_j + (\delta_\$^{(0)} - \delta_j^{(0)}), \tag{C.1}$$

$$\mathbf{e}_{F+M+j}'\boldsymbol{\Phi}_{3\bullet}^Q = \left[(\boldsymbol{\delta}_\$^{(1)} - \boldsymbol{\delta}_j^{(1)})', \mathbf{0}_M', \mathbf{0}_J'\right]. \tag{C.2}$$

Therefore, we have that uncovered interest parity holds under the risk-neutral measure, Q:

$$E_t^Q\Delta s_{j,t+1} = -\frac{1}{2}Var_t\left(\Delta s_{j,t+1}\right) + (r_{\$,t} - r_{j,t}), \tag{C.3}$$

where $-\frac{1}{2}Var_t\left(\Delta s_{j,t+1}\right)$ is a Jensen's inequality term.

C.2 Affine expected rate of depreciation

We now show that in a model the state variables, $\mathbf{x}_t$, follow a VAR(1) process:

$$\mathbf{x}_{t+1} = \boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t + \mathbf{v}_{t+1}$$

where $\mathbf{v}_t \sim iid\ N(0, \boldsymbol{\Sigma})$, and where the stochastic discount factor (SDF) in the domestic and foreign economy are given by

$$\begin{aligned} \xi_{\$,t+1} &= \exp(-r_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}'_{\$,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{\$,t} - \boldsymbol{\lambda}'_{\$,t}\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1}) \\ \xi_{j,t+1} &= \exp(-r_{j,t} - \frac{1}{2}\boldsymbol{\lambda}'_{j,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{j,t} - \boldsymbol{\lambda}'_{j,t}\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1}) \end{aligned}$$

with short rates given by the following affine functions:

$$\begin{aligned} r_{\$,t} &= \delta_{\$}^{(0)} + \boldsymbol{\delta}'_{\$}\mathbf{x}_t \\ r_{j,t} &= \delta_{j}^{(0)} + \boldsymbol{\delta}'_{j}\mathbf{x}_t \end{aligned}$$

and prices of risk given by:

$$\begin{aligned} \boldsymbol{\lambda}_{\$,t} &= \boldsymbol{\lambda}_{\$}^{(0)} + \boldsymbol{\lambda}_{\$}\mathbf{x}_t \\ \boldsymbol{\lambda}_{j,t} &= \boldsymbol{\lambda}_{j}^{(0)} + \boldsymbol{\lambda}_{j}\mathbf{x}_t \end{aligned}$$

then if the expected rate of depreciation is affine in the set of factors:

$$\Delta s_{j,t} = \gamma_j^{(0)} + \boldsymbol{\gamma}'_j\mathbf{x}_t,$$

the rate of depreciation has to be the difference between the log SDFs in the two countries:

$$\Delta s_{j,t+1} = \log \xi_{j,t+1} - \log \xi_{\$,t+1}. \tag{C.4}$$

We note that when the rate of depreciation is not affine in the factors, an additional assumption of market completeness is needed for equation (C.4) to be a sufficient and necessary condition for exchange rate determination (see Backus, Foresi and Telmer, 2001).

The proof is similar to the one in Anderson, Han and Ramazani (2010) and has three steps. First, since the price of a foreign one-period zero-coupon bond, $P_{j,t}^{(1)} = e^{-r_{j,t}}$, must satisfy:

$$E_t\left(\xi_{\$,t+1}e^{\Delta s_{j,t+1}} \times 1/P_{j,t}^{(1)}\right) = E_t\left(\xi_{\$,t+1}e^{\Delta s_{j,t+1}}e^{r_{j,t}}\right) = 1,$$

we have that the uncovered interest parity hypothesis must hold under the (domestic) risk neutral measure:

$$\gamma_j^{(0)} + \boldsymbol{\gamma}'_j\boldsymbol{\mu}^Q = \frac{1}{2}\boldsymbol{\gamma}'_j\boldsymbol{\Sigma}\boldsymbol{\gamma}_j + \delta_{\$}^{(0)} - \delta_j^{(0)} \tag{C.5}$$

$$\boldsymbol{\gamma}'_j\boldsymbol{\Phi}^Q = \boldsymbol{\delta}'_{\$} - \boldsymbol{\delta}'_j \tag{C.6}$$

where $\boldsymbol{\mu}^Q = \boldsymbol{\mu} - \boldsymbol{\lambda}_{\$}^{(0)}$, and $\boldsymbol{\Phi}^Q = \boldsymbol{\Phi} - \boldsymbol{\lambda}_j^{(0)}$.

Second, consider a foreign asset with payoff, $R_{j,t+1}$, at time $t+1$ in units of foreign currency:

$$R_{j,t+1} = \exp\left(r_{j,t} + \frac{1}{2}\boldsymbol{\lambda}'_{j,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{j,t} + \boldsymbol{\lambda}'_{j,t}\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1} \right).$$

with price given by $E_t\left(m_{j,t+1}R_{j,t+1}\right) = E_t\left[\exp(0)\right] = 1$, which implies that $R_{j,t+1}$ is a gross return. Thus, it has to be the case that, when priced by the domestic investor

$$E_t\left(\xi_{\$,t+1}e^{\Delta s_{j,t+1}}R_{j,t+1}\right) = 1.$$

Substituting the law of motion for the rate of depreciation, the domestic SDF, and the expression for the return $R_{j,t+1}$ in the previous expression we get:

$$E_t\left\{\exp\left[-r_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}'_{\$,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{\$,t} - \boldsymbol{\lambda}'_{\$,t}\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1} + \gamma_j^{(0)} + \boldsymbol{\gamma}'_j(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t + \mathbf{v}_{t+1}) \right.\right.$$
$$\left.\left. + r_{j,t} + \frac{1}{2}\boldsymbol{\lambda}'_{j,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{j,t} + \boldsymbol{\lambda}'_{j,t}\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1}\right]\right\} = 1$$
$$E_t\left\{\exp\left[(r_{j,t} - r_{\$,t}) - \frac{1}{2}\boldsymbol{\lambda}'_{\$,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{\$,t} + \frac{1}{2}\boldsymbol{\lambda}'_{j,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{j,t} + \gamma_j^{(0)} + \boldsymbol{\gamma}'_j(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t) \right.\right.$$
$$\left.\left. + \left(\boldsymbol{\lambda}'_{j,t}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\lambda}'_{\$,t}\boldsymbol{\Sigma}^{-1} + \boldsymbol{\gamma}'_j\right)\mathbf{v}_{t+1}\right]\right\} = 1$$

Note that the last term in the previous equation satisfies that

$$E_t\left\{\exp\left[\left(\boldsymbol{\lambda}'_{j,t}\boldsymbol{\Sigma}^{-1} - \boldsymbol{\lambda}'_{\$,t}\boldsymbol{\Sigma}^{-1} + \boldsymbol{\gamma}'_j\right)\mathbf{v}_{t+1}\right]\right\} =$$
$$\exp\left[\frac{1}{2}\left(\boldsymbol{\lambda}_{j,t} - \boldsymbol{\lambda}_{\$,t} + \boldsymbol{\Sigma}\boldsymbol{\gamma}_j\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\lambda}_{j,t} - \boldsymbol{\lambda}_{\$,t} + \boldsymbol{\Sigma}\boldsymbol{\gamma}_j\right)\right] =$$
$$\exp\left(\frac{1}{2}\boldsymbol{\lambda}'_{\$,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{\$,t} + \frac{1}{2}\boldsymbol{\lambda}'_{j,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{j,t} + \frac{1}{2}\boldsymbol{\gamma}'_j\boldsymbol{\Sigma}\boldsymbol{\gamma}_j + \boldsymbol{\lambda}'_{j,t}\boldsymbol{\gamma}_j - \boldsymbol{\lambda}'_{\$,t}\boldsymbol{\gamma}_j - \boldsymbol{\lambda}'_{\$,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{j,t}\right)$$

Thus we have:

$$\exp\left[(r_{j,t} - r_{\$,t}) + \boldsymbol{\lambda}'_{j,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{j,t} + \gamma_j^{(0)} + \boldsymbol{\gamma}'_j(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t) + \frac{1}{2}\boldsymbol{\gamma}'_j\boldsymbol{\Sigma}\boldsymbol{\gamma}_j \right.$$
$$\left. + \boldsymbol{\lambda}'_{j,t}\boldsymbol{\gamma}_j - \boldsymbol{\lambda}'_{\$,t}\boldsymbol{\gamma}_j - \boldsymbol{\lambda}'_{\$,t}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{j,t}\right] = 1$$

Substituting the expressions for the short rates and the prices of risk and expanding, it can be shown that the previous equation has the following form:

$$\exp\left(A + \mathbf{B}'\mathbf{x}_t + \mathbf{x}'_t\mathbf{C}\mathbf{x}_t\right) = 1$$

which implies that $A = 0$, $\mathbf{B} = \mathbf{0}_{n\times 1}$ and $\mathbf{C} = \mathbf{0}_{n\times n}$ where n is the number of factors in order to $A + \mathbf{B}'\mathbf{x}_t + \mathbf{x}'_t\mathbf{C}\mathbf{x}_t = 0$ for all $\mathbf{x}_t$. In particular, since

$$\mathbf{C} = (\boldsymbol{\lambda}_j - \boldsymbol{\lambda}_\$)'\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_j,$$

the quadratic term is equal to zero ($\mathbf{C} = \mathbf{0}_{n\times n}$) when

$$\boldsymbol{\lambda}_j = \boldsymbol{\lambda}_\$ = \boldsymbol{\lambda}. \tag{C.7}$$

As for the linear term, we have that

$$\begin{aligned}
\mathbf{B}' &= \boldsymbol{\delta}_j' - \boldsymbol{\delta}_\$' + \boldsymbol{\gamma}_j'\boldsymbol{\Phi} + \left(\boldsymbol{\lambda}_j^{(0)} - \boldsymbol{\lambda}_\$^{(0)}\right)' \boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda} \\
\mathbf{B}' &= \boldsymbol{\gamma}_j'\left(\boldsymbol{\Phi} - \boldsymbol{\Phi}^Q\right) - \left(\boldsymbol{\lambda}_j^{(0)} - \boldsymbol{\lambda}_\$^{(0)}\right)' \boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda} \\
\mathbf{B}' &= \boldsymbol{\gamma}_j'\boldsymbol{\lambda} - \left(\boldsymbol{\lambda}_j^{(0)} - \boldsymbol{\lambda}_\$^{(0)}\right)' \boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda} \\
\mathbf{B}' &= \left[\boldsymbol{\gamma}_j' - \left(\boldsymbol{\lambda}_j^{(0)} - \boldsymbol{\lambda}_\$^{(0)}\right)' \boldsymbol{\Sigma}^{-1}\right]\boldsymbol{\lambda}
\end{aligned}$$

where the second line comes from the fact that uncovered interest parity is satisfied under Q. Thus the linear term is equal to zero ($\mathbf{B} = \mathbf{0}_{n\times 1}$) when

$$\boldsymbol{\lambda}_j^{(0)} = \boldsymbol{\lambda}_\$^{(0)} - \boldsymbol{\Sigma}\boldsymbol{\gamma}_j. \tag{C.8}$$

Finally, it is possible to show that the constant term:

$$A = (\delta_j^{(0)} - \delta_\$^{(0)}) + \gamma_j^{(0)} + \boldsymbol{\gamma}_j'\boldsymbol{\mu} + (\boldsymbol{\lambda}_j^{(0)} - \boldsymbol{\lambda}_\$^{(0)})'\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_j^{(0)} + (\boldsymbol{\lambda}_j^{(0)} - \boldsymbol{\lambda}_\$^{(0)})'\boldsymbol{\gamma}_j + \frac{1}{2}\boldsymbol{\gamma}_j'\boldsymbol{\Sigma}\boldsymbol{\gamma}_j$$

is equal to zero ($A = 0$) under (C.5), (C.6), (C.7) and (C.8).

Third, we show that given this set of restrictions, the rate of depreciation is given by the difference between the log SDFs between the two countries. In particular, we have that

$$\Delta s_{j,t+1} = \gamma_j^{(0)} + \boldsymbol{\gamma}_j'\mathbf{x}_{t+1} = \gamma_j^{(0)} + \boldsymbol{\gamma}_j'(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t + \mathbf{v}_{t+1}).$$

Substituting (C.5) and (C.6) into the previous equation, we arrive at

$$\begin{aligned}
\Delta s_{j,t+1} &= (\delta_\$^{(0)} - \delta_j^{(0)}) + (\boldsymbol{\delta}_\$ - \boldsymbol{\delta}_j)'\mathbf{x}_t + \boldsymbol{\gamma}_j'(\boldsymbol{\mu} - \boldsymbol{\mu}^Q) + \boldsymbol{\gamma}_j'\left(\boldsymbol{\Phi} - \boldsymbol{\Phi}^Q\right)\mathbf{x}_t - \frac{1}{2}\boldsymbol{\gamma}_j'\boldsymbol{\Sigma}\boldsymbol{\gamma}_j + \boldsymbol{\gamma}_j'\mathbf{v}_{t+1} \\
&= (r_{\$,t} - r_{j,t}) + \boldsymbol{\gamma}_j'(\boldsymbol{\lambda}_\$^{(0)} + \boldsymbol{\lambda}\mathbf{x}_t) - \frac{1}{2}\boldsymbol{\gamma}_j'\boldsymbol{\Sigma}\boldsymbol{\gamma}_j + \boldsymbol{\gamma}_j'\mathbf{v}_{t+1}
\end{aligned}$$

Note however that equation (C.8) implies that

$$\begin{gathered}
\boldsymbol{\gamma}_j'(\boldsymbol{\lambda}_\$^{(0)} + \boldsymbol{\lambda}\mathbf{x}_t) - \frac{1}{2}\boldsymbol{\gamma}_j'\boldsymbol{\Sigma}\boldsymbol{\gamma}_j = \\
(\boldsymbol{\lambda}_\$^{(0)} - \boldsymbol{\lambda}_j^{(0)})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\lambda}_\$^{(0)} + \boldsymbol{\lambda}\mathbf{x}_t) - \frac{1}{2}(\boldsymbol{\lambda}_\$^{(0)} - \boldsymbol{\lambda}_j^{(0)})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\lambda}_\$^{(0)} - \boldsymbol{\lambda}_j^{(0)}) = \\
\frac{1}{2}\left(\boldsymbol{\lambda}_\$^{(0)} + \boldsymbol{\lambda}\mathbf{x}_t\right)\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\lambda}_\$^{(0)} + \boldsymbol{\lambda}\mathbf{x}_t\right) - \frac{1}{2}\left(\boldsymbol{\lambda}_j^{(0)} + \boldsymbol{\lambda}\mathbf{x}_t\right)\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\lambda}_j^{(0)} + \boldsymbol{\lambda}\mathbf{x}_t\right) = \\
\frac{1}{2}\boldsymbol{\lambda}_{\$,t}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}_{j,t}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{j,t}
\end{gathered}$$

and that

$$\begin{aligned}
\boldsymbol{\gamma}_j'\mathbf{v}_{t+1} &= (\boldsymbol{\lambda}_\$^{(0)} - \boldsymbol{\lambda}_j^{(0)})'\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1} \\
&= (\boldsymbol{\lambda}_{\$,t} - \boldsymbol{\lambda}_{j,t})'\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1}
\end{aligned}$$

Thus we have that:

$$\begin{aligned}
\Delta s_{j,t+1} &= \left(-r_{j,t} - \frac{1}{2}\boldsymbol{\lambda}_{j,t}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{j,t} - \boldsymbol{\lambda}_{j,t}'\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1}\right) - \left(-r_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}_{\$,t}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_{\$,t} - \boldsymbol{\lambda}_{\$,t}'\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1}\right) \\
&= \log m_{j,t+1} - \log m_{\$,t+1}
\end{aligned}$$

which is equation (11) in the main text.

C.3 Obtaining the foreign SDF

As noted in the previous section, when the rate of depreciation is affine in the set of pricing factors (which, in our case is trivially satisfied given that $\Delta s_{j,t+1}$ is itself a pricing factor), the law of one price tells us that one of the numeraire SDF, the country j SDF and the rate of depreciation of the currency j is redundant and can be constructed from the other two. In particular, we can solve for the foreign SDF:

$$\log \xi_{j,t+1} = \Delta s_{j,t+1} + \log \xi_{\$,t+1}$$

and substituting the law of motion for the rate of depreciation (equation 5 in the main text) and the domestic SDF (equation 6 in the main text) into this equation and rearranging we obtain:

$$\begin{aligned}
\log \xi_{j,t+1} &= \mathbf{e}'_{F+M+j}(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t + \mathbf{v}_{t+1}) - r_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}'_t\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_t - \boldsymbol{\lambda}'_t\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1} \\
&= \mathbf{e}'_{F+M+j}(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t) - r_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}'_t\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_t - \left(\boldsymbol{\lambda}'_t - \mathbf{e}'_{F+M+j}\boldsymbol{\Sigma}\right)\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1}
\end{aligned}$$

Now define $\boldsymbol{\lambda}_t^{(j)\prime} \equiv \boldsymbol{\lambda}'_t - \mathbf{e}'_{F+M+j}\boldsymbol{\Sigma}$ and use this expression to substitute $\boldsymbol{\lambda}_t$ into the previous equation:

$$\begin{aligned}
\log m_{j,t+1} &= \mathbf{e}'_{F+M+j}(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t) - r_{\$,t} - \left(\boldsymbol{\lambda}_t^{(j)} - \boldsymbol{\Sigma}\mathbf{e}_{F+M+j}\right)'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\lambda}_t^{(j)} - \boldsymbol{\Sigma}\mathbf{e}_{F+M+j}\right) \\
&\quad -\boldsymbol{\lambda}_t^{(j)}\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1} \\
&= \mathbf{e}'_{F+M+j}(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t) - r_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}_t^{(j)\prime}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_t^{(j)} - \mathbf{e}'_{F+M+j}\boldsymbol{\lambda}_t^{(j)} \\
&\quad -\frac{1}{2}\mathbf{e}'_{F+M+j}\boldsymbol{\Sigma}\mathbf{e}_{F+M+j} - \boldsymbol{\lambda}_t^{(j)}\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1} \\
&= \mathbf{e}'_{F+M+j}(\boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t) - r_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}_t^{(j)\prime}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_t^{(j)} - \mathbf{e}'_{F+M+j}\left(\boldsymbol{\lambda}_t - \boldsymbol{\Sigma}\mathbf{e}_{F+M+j}\right) \\
&\quad -\frac{1}{2}\mathbf{e}'_{F+M+j}\boldsymbol{\Sigma}\mathbf{e}_{F+M+j} - \boldsymbol{\lambda}_t^{(j)}\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1} \\
&= \mathbf{e}'_{F+M+j}\left[(\boldsymbol{\mu} - \boldsymbol{\lambda}_0) + (\boldsymbol{\Phi} - \boldsymbol{\lambda}_1)\mathbf{x}_t\right] - r_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}_t^{(j)\prime}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_t^{(j)} \\
&\quad +\frac{1}{2}\mathbf{e}'_{F+M+j}\boldsymbol{\Sigma}\mathbf{e}_{F+M+j} - \boldsymbol{\lambda}_t^{(j)}\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1} \\
&= \mathbf{e}'_{F+M+j}(\boldsymbol{\mu}^Q + \boldsymbol{\Phi}^Q\mathbf{x}_t) - r_{\$,t} - \frac{1}{2}\boldsymbol{\lambda}_t^{(j)\prime}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_t^{(j)} + \frac{1}{2}\mathbf{e}'_{F+M+j}\boldsymbol{\Sigma}^{-1}\mathbf{e}_{F+M+j} - \boldsymbol{\lambda}_t^{(j)}\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1}
\end{aligned}$$

Note however that the first term in the previous expression satisfies that:

$$\mathbf{e}'_{F+M+j}(\boldsymbol{\mu}^Q + \boldsymbol{\Phi}^Q\mathbf{x}_t) = E_t^Q\Delta s_{j,t+1} = -\frac{1}{2}\mathbf{e}'_{F+M+j}\boldsymbol{\Sigma}\mathbf{e}_{F+M+j} + (r_{\$,t} - r_{j,t})$$

Thus we have that:

$$m_{j,t+1} = \exp\left(-r_{j,t} - \frac{1}{2}\boldsymbol{\lambda}_t^{(j)\prime}\boldsymbol{\Sigma}^{-1}\boldsymbol{\lambda}_t^{(j)} - \boldsymbol{\lambda}_t^{(j)}\boldsymbol{\Sigma}^{-1}\mathbf{v}_{t+1}\right)$$

D Invariant transformations of multi-country term structure models

Assume the following multi-country term structure model:

$$\begin{aligned}\mathbf{r}_t &= \boldsymbol{\Delta}_0 + \boldsymbol{\Delta}_1 \mathbf{x}_t \\ \mathbf{x}_{t+1} &= \boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t + \mathbf{v}_{t+1} \\ \mathbf{x}_{t+1} &= \boldsymbol{\mu}^Q + \boldsymbol{\Phi}^Q\mathbf{x}_t + \mathbf{v}^Q_{t+1}\end{aligned}$$

where both $\mathbf{v}_t$ and $\mathbf{v}^Q_t$ are *iid* $N(0, \boldsymbol{\Sigma})$, and $\mathbf{x}_t = (\mathbf{x}'_{1,t}, \mathbf{x}'_{2,t})'$ being $\mathbf{x}_{1,t}$ a latent set of factors, and $\mathbf{x}_{2,t}$ observable. As in Dai and Singleton (2000), we interested in applying invariant transformations, $\widehat{\mathbf{x}}_t = \mathbf{c} + \mathbf{D}\mathbf{x}_t$. We then have that the model model above is observationally equivalent to:

$$\begin{aligned}\mathbf{r}_t &= \widehat{\boldsymbol{\Delta}}_0 + \widehat{\boldsymbol{\Delta}}_1 \mathbf{x}_t \\ \mathbf{x}_{t+1} &= \widehat{\boldsymbol{\mu}} + \widehat{\boldsymbol{\Phi}}\mathbf{x}_t + \widehat{\mathbf{v}}_{t+1} \\ \mathbf{x}_{t+1} &= \widehat{\boldsymbol{\mu}}^Q + \widehat{\boldsymbol{\Phi}}^Q\mathbf{x}_t + \widehat{\mathbf{v}}^Q_{t+1}\end{aligned}$$

where now both $\widehat{\mathbf{v}}_t$ and $\widehat{\mathbf{v}}^Q_t$ are *iid* $N(0, \widehat{\boldsymbol{\Sigma}})$ and

$$\begin{aligned}\widehat{\boldsymbol{\Delta}}_0 &= \boldsymbol{\Delta}_0 - \boldsymbol{\Delta}_1\mathbf{D}^{-1}\mathbf{c} \\ \widehat{\boldsymbol{\Delta}}_1 &= \boldsymbol{\Delta}_1\mathbf{D}^{-1} \\ \widehat{\boldsymbol{\mu}} &= (\mathbf{I} - \mathbf{D}\boldsymbol{\Phi}\mathbf{D}^{-1})\mathbf{c} + \mathbf{D}\boldsymbol{\mu} \\ \widehat{\boldsymbol{\Phi}} &= \mathbf{D}\boldsymbol{\Phi}\mathbf{D}^{-1} \\ \widehat{\boldsymbol{\mu}}^Q &= (\mathbf{I} - \mathbf{D}\boldsymbol{\Phi}^Q\mathbf{D}^{-1})\mathbf{c} + \mathbf{D}\boldsymbol{\mu}^Q \\ \widehat{\boldsymbol{\Phi}}^Q &= \mathbf{D}\boldsymbol{\Phi}^Q\mathbf{D}^{-1} \\ \widehat{\boldsymbol{\Sigma}} &= \mathbf{D}\boldsymbol{\Sigma}\mathbf{D}'\end{aligned}$$

Of special interest to us are those invariant transformations that leave the set of observable variables, $\mathbf{x}_{2,t}$, unchanged. Such transformations can be expressed the following way:

$$\begin{pmatrix}\widehat{\mathbf{x}}_{1,t} \\ \widehat{\mathbf{x}}_{2,t}\end{pmatrix} = \begin{pmatrix}\mathbf{c}_1 \\ \mathbf{0}\end{pmatrix} + \begin{pmatrix}\mathbf{D}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{x}_{1,t} \\ \mathbf{x}_{2,t}\end{pmatrix} = \begin{pmatrix}\mathbf{c}_1 + \mathbf{D}_1\mathbf{x}_{1,t} \\ \mathbf{x}_{2,t}\end{pmatrix}$$

E Proofs

E.1 Proof of Lemma 1

To proof this lemma, we use the invariant transformations of multi-country term structure models above. In particular, we need to focus on invariant transformations that leave the set of macro variables and exchange rates unchanged:

$$\begin{pmatrix}\widehat{\mathbf{f}}_t \\ \mathbf{m}_t \\ \Delta\mathbf{s}_t\end{pmatrix} = \begin{pmatrix}\mathbf{c}_1 \\ \mathbf{0} \\ \mathbf{0}\end{pmatrix} + \begin{pmatrix}\mathbf{D}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_J\end{pmatrix}\begin{pmatrix}\mathbf{f}_t \\ \mathbf{m}_t \\ \Delta\mathbf{s}_t\end{pmatrix}.$$

For simplicity, we assume that $\mathbf{\Phi}_{11}^{Q}$ can be diagonalized, that is $\mathbf{\Phi}_{11}^{Q} = \mathbf{T}\mathbf{\Lambda}\mathbf{T}^{-1}$ where $\mathbf{\Lambda}$ is a diagonal matrix that contains the eigenvalues of $\mathbf{\Phi}_{11}^{Q}$, and $\mathbf{P}$ is a matrix that contains the corresponding eigenvectors. The following two invariant transformations deliver the model in Lemma 1. First, we apply:

$$\begin{pmatrix} \widehat{\mathbf{f}}_t \\ \mathbf{m}_t \\ \Delta\mathbf{s}_t \end{pmatrix} = \begin{pmatrix} (\mathbf{I}-\mathbf{\Lambda})^{-1}(\mathbf{E}\mathbf{k} - \mathbf{T}^{-1}\boldsymbol{\mu}_1^{Q}) \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{T}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_J \end{pmatrix} \begin{pmatrix} \mathbf{f}_t \\ \mathbf{m}_t \\ \Delta\mathbf{s}_t \end{pmatrix}.$$

where

$$\mathbf{E} = \begin{pmatrix} \mathbf{I}_{J+1} \\ \mathbf{0} \end{pmatrix}$$

and $\mathbf{k}$ is a $(J+1)$ dimensional vector.

Second, we exploit that for a given diagonal matrix such as $\mathbf{\Lambda}$, we can pre- and post-multiply it by another diagonal matrix, $\mathbf{B}$, and leave it unchanged it: $\mathbf{\Lambda} = \mathbf{L}\mathbf{\Lambda}\mathbf{L}^{-1}$. In particular using

$$\begin{pmatrix} \widetilde{\mathbf{f}}_t \\ \mathbf{m}_t \\ \Delta\mathbf{s}_t \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{L} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_J \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{f}}_t \\ \mathbf{m}_t \\ \Delta\mathbf{s}_t \end{pmatrix}$$

where

$$\mathbf{L} = \begin{pmatrix} \sum_{j=0}^{J} \widehat{\delta}_{1j} & 0 & \cdots & 0 \\ 0 & \sum_{j=0}^{J} \widehat{\delta}_{2j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j=0}^{J} \widehat{\delta}_{Fj} \end{pmatrix}$$

where $\widehat{\delta}_{ij}$ is the i-th element of vector $\widehat{\boldsymbol{\delta}}_j^{(1)}$, the vector of factor loadings of the short-rate obtained from the first invariant transformation. Under such transformation, the factor loadings for the short-rate will sum up to one, and thus the model can be expressed in the canonical form of Lemma 1 with $\mathbf{\Psi}_{11}^{Q} = \mathbf{\Lambda}$ and $\mathbf{k}_{\infty}^{Q} = \mathbf{k}$.

E.2 Proof of Proposition 2

As noted in the main text, the multi-country term structure model implies yields on domestic and foreign zero coupon bonds that are affine functions of the bond state variables $\mathbf{f}_t$:

$$\mathbf{y}_t = \mathbf{a}_f + \mathbf{b}_f \mathbf{f}_t \tag{E.1}$$

where $\mathbf{y}_t = (y_{\$,t}^{(1)}, \ldots, y_{\$,t}^{(N)}, \ldots, y_{j,t}^{(n)}, \ldots, y_{J,t}^{(N)})'$ and the corresponding elements of $\mathbf{a}_f$ and $\mathbf{b}_f$ are computed using the system of recursive relations in the main text. If we choose to work with "bond" state variables that are linear combinations of the yields themselves, $\mathbf{f}_t = \mathbf{P}'\mathbf{y}_t$, where $\mathbf{P}$ is a full-rank matrix of weights, then, by equation (E.1) we have:

$$\mathbf{f}_t = \mathbf{P}'\mathbf{y}_t = \mathbf{P}'(\mathbf{a}_f + \mathbf{b}_f \mathbf{f}_t),$$

which implies that our model will only be self-consistent when $\mathbf{P}'\mathbf{a}_f = \mathbf{0}_{F\times 1}$ and $\mathbf{P}'\mathbf{b}_f = \mathbf{I}_F$.

On the other hand, the canonical multi-country dynamic term structure model in Lemma 1 implies yields on domestic and foreign zero coupon bonds that are affine in $\mathbf{z}_t$:

$$\mathbf{y}_t = \mathbf{a}_z + \mathbf{b}_z \mathbf{z}_t \tag{E.2}$$

As such, bond state variables are that linear combinations of yields are simply (invariant) affine transformations of the latent factors $\mathbf{z}_t$:

$$\begin{aligned} \mathbf{f}_t &= \mathbf{P}'\mathbf{y}_t \\ &= \mathbf{P}'(\mathbf{a}_z + \mathbf{b}_z \mathbf{z}_t) \\ &= \mathbf{c} + \mathbf{D}\mathbf{z}_t \end{aligned} \tag{E.3}$$

Thus, we can apply the results on invariant transformations above with

$$\begin{pmatrix} \mathbf{f}_t \\ \mathbf{m}_t \\ \Delta\mathbf{s}_t \end{pmatrix} = \begin{pmatrix} \mathbf{P}'\mathbf{a}_z \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{P}'\mathbf{b}_z & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_M & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_J \end{pmatrix} \begin{pmatrix} \mathbf{z}_t \\ \mathbf{m}_t \\ \Delta\mathbf{s}_t \end{pmatrix}$$

to obtain an observationally equivalent model such that:

$$\mathbf{y}_t = \mathbf{a}_z + \mathbf{b}_z \mathbf{z}_t = \mathbf{a}_f + \mathbf{b}_f \mathbf{f}_t$$

Substituting (E.3) into the previous expression:

$$\begin{aligned} \mathbf{a}_z + \mathbf{b}_z \mathbf{z}_t &= \mathbf{a}_f + \mathbf{b}_f \mathbf{f}_t \\ &= \mathbf{a}_f + \mathbf{b}_f \mathbf{P}'\mathbf{y}_t \\ &= \mathbf{a}_f + \mathbf{b}_f \left(\mathbf{c} + \mathbf{D}\mathbf{z}_t\right) \\ &= \mathbf{a}_f + \mathbf{b}_f \mathbf{c} + \mathbf{b}_f \mathbf{D}\mathbf{z}_t \end{aligned}$$

Thus, we have that

$$\begin{aligned} \mathbf{b}_z &= \mathbf{b}_f \mathbf{D} \\ \mathbf{a}_z &= \mathbf{a}_f + \mathbf{b}_f \mathbf{c} \end{aligned}$$

Finally, premultiplying by $\mathbf{P}'$, and solving for $\mathbf{P}'\mathbf{b}_f$ in the first equation

$$\mathbf{P}'\mathbf{b}_f = (\mathbf{P}'\mathbf{b}_z)\mathbf{D}^{-1} = \mathbf{I}$$

and solving for $\mathbf{P}'\mathbf{a}_f$ in the second equation

$$\mathbf{P}'\mathbf{a}_f = \mathbf{P}'\mathbf{a}_z - (\mathbf{P}'\mathbf{b}_f)\mathbf{c} = \mathbf{0}$$

we obtain that the corresponding conditions for self-consistency are satisfied.

F Additional details on Step 1: Fitting yields

In this section, we provide additional details on the estimation of the parameters driving the risk-neutral dynamics of the bond factors. In particular, we estimate the parameters under Q directly by minimizing the sum (across maturities, countries, and time) of the squared differences between model predictions and actual yields:

$$\min_{\boldsymbol{\mu}_1^Q, \boldsymbol{\Phi}_{11}^Q, \boldsymbol{\Delta}^{(0)}, \boldsymbol{\Delta}^{(1)}} \sum_{n=1}^{N} \sum_{j=1}^{J+1} \sum_{t=1}^{T} (y_{j,t}^{(n)} - a_j^{(n)} - \mathbf{b}_j^{(n)\prime} \mathbf{f}_t)^2. \tag{F.1}$$

subject to the self-consistency restrictions in Proposition 2 in the main text. As JPS, we focus on the case where the eigenvalues in $\boldsymbol{\Psi}_{11}^Q$ are real and distinct. In order to satisfy Hamilton and Wu's (2012) identification restriction (see Remark 1 of Lemma 1), we assume that the eigenvalues of $\boldsymbol{\Psi}_{11}^Q$ are distributed according to a power law relation:

$$\psi_{11,j}^Q = \overline{\psi}_{11}^Q \varphi^{j-1}, \qquad j = 1, \ldots, F \tag{F.2}$$

where $\overline{\psi}_{11}^Q$ is the largest eigenvalue of $\boldsymbol{\Psi}_{11}^Q$ and the power scaling coefficient $0 < \varphi < 1$ controls the spacing between different eigenvalues. We refer the reader to Calvet, Fisher and Wu (2010) for an application of power law structures to term structure modeling. Moreover, as in Christensen, Diebold and Rudebusch (2010), we set $\overline{\psi}_{11}^Q = 1.00$ in order to replicate the level factor that characterizes the international cross-section of interest rates. In this way, we reduce the number of free parameters to 29 and we estimate them by minimizing (F.1) directly.

We estimate the power scaling coefficient, φ, sequentially through concentration. That is, for a given value of φ, we numerically minimize the sum of the squared differences between model predictions and actual yields as a function of the rest of parameters driving the risk-neutral measure. We then search over the possible values of φ for the one that minimizes the sum of squared differences to get our estimate of this parameter.

Below, we show that it is possible to concentrate out $\mathbf{k}_\infty^Q$ which further reduces the number of free parameters to be estimated directly in the minimization of (F.1).

We also employ a score algorithm to minimize the sum of squared differences between actual and model-implied yields, with analytical expressions for the gradient and the expected value of the Hessian of the criterion function. By providing both the gradient and an estimate of the Hessian of the criterion function, we obtain a very fast convergence of our optimization algorithm (e.g., around one minute for an eight factor and four country model).

Finally, we note that we could have chosen to work with linear combinations of yields that resemble empirical measures of level, slope and curvature. For example, we could define the level as the 10-year yield, the slope as the difference between the 10- and 1-year yields, and the curvature as twice the 5-year yield minus the sum of the 1- and 10-year yields. However, in our estimation below we assume that the bond state factors are priced perfectly by our model. Thus, by using principal components, we account for as much of the variability in the international cross-section of yields as possible, which, in turns, greatly minimizes the pricing errors of the model with respect to any other linear combination of yields that could potentially be used.

F.1 Concentrating $\mathbf{k}_\infty^Q$ out

In order to concentrate $\mathbf{k}_\infty^Q$ out of the sum of squared residuals, first note that the canonical specification of our multi-country term structure model implies:

$$y_{j,t}^{(n)} = a_{j,z}^{(n)} + \mathbf{b}_{j,z}^{(n)}\mathbf{z}_t \tag{F.3}$$

where $b_{j,z}^{(n)}$ does not depend on $\mathbf{k}_\infty^Q$ and

$$\begin{aligned} a_{j,z}^{(n)} &= \boldsymbol{\alpha}_{j,z}\mathbf{k}_\infty^Q + \boldsymbol{\beta}_{j,z}\boldsymbol{\Omega}_{13}\mathbf{e}_j + \boldsymbol{\gamma}_{j,z}vec(\boldsymbol{\Omega}_{11}) \qquad \text{for } j = 1, ..., J \\ a_{j,z}^{(n)} &= \boldsymbol{\alpha}_{j,z}\mathbf{k}_\infty^Q + \boldsymbol{\gamma}_{j,z}vec(\boldsymbol{\Omega}_{11}) \qquad \text{for } j = \$ \end{aligned}$$

where

$$\boldsymbol{\beta}_{j,z} = \begin{pmatrix} 0 \\ \frac{1}{2}\mathbf{b}_{j,z}^{(1)} \\ \vdots \\ \frac{1}{n}\sum_{i=1}^{n-1} i\mathbf{b}_{j,z}^{(i)} \end{pmatrix} \qquad \boldsymbol{\gamma}_{j,z} = \begin{bmatrix} 0 \\ -\frac{1}{2} \times \frac{1}{2}\left(\mathbf{b}_{j,z}^{(1)\prime} \otimes \mathbf{b}_{j,z}^{(1)\prime}\right) \\ \vdots \\ -\frac{1}{n} \times \frac{1}{2}\sum_{i=1}^{n-1} i^2\left(\mathbf{b}_{j,z}^{(i)\prime} \otimes \mathbf{b}_{j,z}^{(i)\prime}\right) \end{bmatrix}$$

and $\boldsymbol{\alpha}_{j,z} = \boldsymbol{\beta}_{j,z}\mathbf{E}$ with

$$\mathbf{E} = \begin{pmatrix} \mathbf{I}_{J+1} \\ \mathbf{0} \end{pmatrix}$$

Second, stacking the pricing equations across countries and maturities, we obtain:

$$\mathbf{y}_t = \mathbf{a}_z + \mathbf{b}_z\mathbf{z}_t \tag{F.4}$$

where, again, $\mathbf{b}_z$ does not depend on $\mathbf{k}_\infty^Q$ and

$$\mathbf{a}_z = \boldsymbol{\alpha}_z\mathbf{k}_\infty^Q + \boldsymbol{\beta}_z\widetilde{\boldsymbol{\theta}}^Q + \boldsymbol{\gamma}_z vec(\boldsymbol{\Omega}_{11}) \tag{F.5}$$

with

$$\boldsymbol{\alpha}_z = \begin{pmatrix} \boldsymbol{\alpha}_{\$,z} \\ \boldsymbol{\alpha}_{1,z} \\ \vdots \\ \boldsymbol{\alpha}_{J,z} \end{pmatrix} \qquad \boldsymbol{\beta}_z = \begin{pmatrix} \boldsymbol{\beta}_{\$,z} & 0 & \cdots & 0 \\ 0 & \boldsymbol{\beta}_{1,z} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\beta}_{J,z} \end{pmatrix} \qquad \boldsymbol{\gamma}_z = \begin{pmatrix} \boldsymbol{\gamma}_{\$,z} \\ \boldsymbol{\gamma}_{1,z} \\ \vdots \\ \boldsymbol{\gamma}_{J,z} \end{pmatrix}$$

and

$$\widetilde{\boldsymbol{\theta}}^Q = \begin{bmatrix} \mathbf{0} \\ vec(\boldsymbol{\Omega}_{13}) \end{bmatrix}$$

Third, rotating the factors to $\mathbf{f}_t = \mathbf{P}'\mathbf{y}_t$, we have that

$$\begin{gathered} \mathbf{f}_t = \mathbf{P}'\mathbf{y}_t = \mathbf{P}'(\mathbf{a}_z + \mathbf{b}_z\mathbf{z}_t) = \mathbf{c} + \mathbf{D}\mathbf{z}_t \\ \boldsymbol{\Sigma}_{13} = \mathbf{D}\boldsymbol{\Omega}_{13} \longrightarrow vec(\boldsymbol{\Omega}_{13}) = (\mathbf{I}_J \otimes \mathbf{D}^{-1})vec(\boldsymbol{\Sigma}_{13}) \\ \boldsymbol{\Sigma}_{11} = \mathbf{D}\boldsymbol{\Omega}_{13}\mathbf{D}' \longrightarrow vec(\boldsymbol{\Omega}_{11}) = (\mathbf{D}^{-1} \otimes \mathbf{D}^{-1})vec(\boldsymbol{\Sigma}_{11}) \\ \widetilde{\boldsymbol{\theta}}^Q = (\mathbf{I}_{J+1} \otimes \mathbf{D}^{-1})\widetilde{\boldsymbol{\mu}}^Q \end{gathered}$$

where

$$\widetilde{\boldsymbol{\mu}}^Q = \begin{bmatrix} \mathbf{0} \\ vec(\boldsymbol{\Sigma}_{13}) \end{bmatrix}$$

Therefore, we can write the following model the bond yields as a function of the new factors $\mathbf{f}_t$:

$$\mathbf{y}_t = \mathbf{a}_f + \mathbf{b}_f \mathbf{f}_t$$

where, $\mathbf{b}_f$ does not depend on $\mathbf{k}_\infty^Q$ and

$$\mathbf{a}_f = \boldsymbol{\alpha}_f \mathbf{k}_\infty^Q + \boldsymbol{\beta}_f \widetilde{\boldsymbol{\mu}}^Q + \boldsymbol{\gamma}_f vec(\boldsymbol{\Sigma}_{11}) \tag{F.6}$$

with

$$\begin{aligned} \mathbf{F} &= \mathbf{I} - (\mathbf{b}_z \mathbf{D}^{-1})\mathbf{P}' \\ \boldsymbol{\alpha}_f &= \mathbf{F}\boldsymbol{\alpha}_z \\ \boldsymbol{\beta}_f &= \mathbf{F}\boldsymbol{\beta}_z(\mathbf{I}_{J+1} \otimes \mathbf{D}^{-1}) \\ \boldsymbol{\gamma}_f &= \mathbf{F}\boldsymbol{\gamma}_z(\mathbf{D}^{-1} \otimes \mathbf{D}^{-1}) \end{aligned}$$

Finally, note that equation (F.6) is linear in $\mathbf{k}_\infty^Q$. Therefore, when solving the first order condition of the optimization problem with respect to $\mathbf{k}_\infty^Q$ we have that

$$\begin{aligned} SSR &= \sum_{t=1}^{T} \mathbf{u}_t' \mathbf{u}_t \\ \frac{\partial SSR}{\partial \mathbf{k}_\infty^Q} &= \sum_{t=1}^{T} \frac{\partial \mathbf{u}_t'}{\partial \mathbf{k}_\infty^Q} \mathbf{u}_t = 0 \end{aligned}$$

where

$$\mathbf{u}_t = \mathbf{y}_t - \boldsymbol{\alpha}_f \mathbf{k}_\infty^Q - \boldsymbol{\beta}_f \widetilde{\boldsymbol{\mu}}^Q - \boldsymbol{\gamma}_f vec(\boldsymbol{\Sigma}_{11}) - \mathbf{b}_f \mathbf{f}_t$$

$$\frac{\partial \mathbf{u}_t'}{\partial \mathbf{k}_\infty^Q} = -\boldsymbol{\alpha}_f'$$

which implies that $\widehat{\mathbf{k}}_\infty^Q$ must satisfy:

$$\widehat{\mathbf{k}}_\infty^Q = (\boldsymbol{\alpha}_f' \boldsymbol{\alpha}_f)^{-1} \left\{ \boldsymbol{\alpha}_f' \left[\overline{\mathbf{y}} - \boldsymbol{\beta}_f \widetilde{\boldsymbol{\mu}}^Q - \boldsymbol{\gamma}_f vec(\boldsymbol{\Sigma}_{11}) - \mathbf{b}_f \overline{\mathbf{f}} \right] \right\}$$

where $\overline{\mathbf{y}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{y}_t$ and $\overline{\mathbf{f}} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{f}_t$.

F.2 Details on the optimization algorithm

To speed up the minimization of the sum (across maturities, countries, and time) of squared differences between actual and model-implied yields, we use a scoring algorithm, which is a Newton-Raphson optimization algorithm where one approxi mates the Hessian of the function to be minimized by its expectation.

In particular, let $f(\boldsymbol{\psi})$ be the function to be minimized

$$f(\boldsymbol{\psi}) = \sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{j=1}^{J+1}(y_{j,t}^{(n)} - a_j^{(n)} - \mathbf{b}_j^{(n)\prime}\mathbf{f}_t)^2 = \sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{j=1}^{J+1}\left(u_{j,t}^{(n)}\right)^2$$

where $\boldsymbol{\psi} = \left[\mathbf{k}_{\infty}^{Q\prime}, vec(\boldsymbol{\Gamma})'\right]'$ is the vector of structural parameters. Then, the relevant elements of the gradient vector and hessian matrix of $f(\boldsymbol{\psi})$ can be obtained from:

$$g_k = \frac{\partial f(\boldsymbol{\psi})}{\partial \psi_k} = 2\sum_{n=1}^{N}\sum_{j=1}^{J+1}\sum_{t=1}^{T}\frac{\partial u_{j,t}^{(n)}}{\partial \psi_k}u_{j,t}^{(n)}$$

$$h_{kl} = \frac{\partial^2 f(\boldsymbol{\psi})}{\partial \psi_k \partial \psi_l} = 2\sum_{n=1}^{N}\sum_{j=1}^{J+1}\sum_{t=1}^{T}\left[\frac{\partial^2 u_{j,t}^{(n)}}{\partial \psi_k \partial \psi_l}u_{j,t}^{(n)} + \frac{\partial u_{j,t}^{(n)}}{\partial \psi_k}\frac{\partial u_{j,t}^{(n)}}{\partial \psi_l}\right]$$

The idea of the scoring algorithm is to replace the true h_{kl} by the approximate hessian that does not depend on second derivatives:

$$\widetilde{h}_{kl} = \frac{\partial^2 f(\boldsymbol{\psi})}{\partial \psi_k \partial \psi_l} = 2\sum_{n=1}^{N}\sum_{j=1}^{J+1}\sum_{t=1}^{T}\left[\frac{\partial u_{j,t}^{(n)}}{\partial \psi_k}\frac{\partial u_{j,t}^{(n)}}{\partial \psi_l}\right]$$

This choice can be justified under the assumption that pricing errors $u_{j,t}^{(n)}$ are orthogonal to the pricing factors, $\mathbf{f}_t$. In such a case, when T is sufficiently large, the first term of h_{kl} vanishes:

$$\sum_{n=1}^{N}\sum_{j=1}^{J+1}\frac{1}{T}\sum_{t=1}^{T}\left[\frac{\partial^2 u_{j,t}^{(n)}}{\partial \psi_k \partial \psi_l}u_{j,t}^{(n)}\right] \to \sum_{n=1}^{N}\sum_{j=1}^{J+1}E\left[\frac{\partial^2 u_{j,t}^{(n)}}{\partial \psi_k \partial \psi_l}u_{j,t}^{(n)}\right] = 0$$

given that $\partial^2 u_{j,t}^{(n)}/\partial \psi_k \partial \psi_l$ is a function of $\mathbf{f}_t$.

In turn, both the gradient and (approximate) hessian of the sum of squared residuals require the analytical derivatives of the pricing errors with respect to the structural parameters:

$$\frac{\partial u_{j,t}^{(n)}}{\partial \psi_k} = -\frac{\partial a_j^{(n)}}{\partial \psi_k} - \frac{\partial \mathbf{b}_j^{(n)\prime}}{\partial \psi_k}\mathbf{f}_t$$

and, thus, the derivatives of the bond price coefficients $a_j^{(n)} = -A_j^{(n)}/n$ and $\mathbf{b}_j^{(n)} = -\mathbf{B}_j^{(n)}/n$ which we can evaluate analytically by an extra set of recursions that run in parallel with the pricing equations. As shown in Diez de los Rios (2010), these extra recursions are obtained by differentiating the pricing equations in (B.1) and (B.2). For example for the case of the numeraire currency:

$$\frac{\partial A_{\$}^{(n)}}{\partial \psi_i} = \frac{\partial A_{\$}^{(n-1)}}{\partial \psi_i} + \frac{\partial \mathbf{B}_{\$}^{(n-1)\prime}}{\partial \psi_i}\boldsymbol{\mu}_1^Q + \mathbf{B}_{\$}^{(n-1)\prime}\frac{\partial \boldsymbol{\mu}_1^Q}{\partial \psi_i} + \frac{\partial \mathbf{B}_{\$}^{(n-1)\prime}}{\partial \psi_i}\boldsymbol{\Sigma}_{11}\mathbf{B}_{\$}^{(n-1)}$$
$$+\frac{1}{2}\mathbf{B}_{\$}^{(n-1)\prime}\frac{\partial \boldsymbol{\Sigma}_{11}}{\partial \psi_i}\mathbf{B}_{\$}^{(n-1)} - \frac{\partial \delta_{\$}^{(0)}}{\partial \psi_i},$$

$$\frac{\partial \mathbf{B}_{\$}^{(n)\prime}}{\partial \psi_i} = \frac{\partial \mathbf{B}_{\$}^{(n-1)\prime}}{\partial \psi_i}\boldsymbol{\Phi}_{11}^Q + \mathbf{B}_{\$}^{(n-1)\prime}\frac{\partial \boldsymbol{\Phi}_{11}^Q}{\partial \psi_i} - \frac{\partial \boldsymbol{\delta}_{\$}^{(1)\prime}}{\partial \psi_i}.$$

with $\partial A_{\$}^{(1)}/\partial \psi_i = -\partial \delta_{\$}^{(0)}/\partial \psi_i$ and $\partial \mathbf{B}_{\$}^{(1)}/\partial \psi_i = -\partial \boldsymbol{\delta}_{\$}^{(1)}/\partial \psi_i$.

G Standard Errors

In this appendix, we provide standard errors for the estimation of the parameters driving the dynamics of the pricing factors under the risk neutral measure, and those driving the price of bond and foreign exchange risk.

Stage 1. Remember that we estimate the parameters governing the risk-neutral distribution Q, $\boldsymbol{\psi} = \left[\mathbf{k}_{\infty}^{Q\prime}, vec(\boldsymbol{\Gamma})'\right]'$, by minimizing the sum (across maturities, countries and time) of the squared differences between model predictions and actual yields:

$$\widehat{\boldsymbol{\psi}} = \arg\min f(\boldsymbol{\psi}) = \arg\min \sum_{t=1}^{T}\sum_{n=1}^{N}\sum_{j=1}^{J+1}\left(u_{j,t}^{(n)}\right)^2$$

where $u_{j,t}^{(n)} = y_{j,t}^{(n)} - a_j^{(n)} - \mathbf{b}_j^{(n)\prime}\mathbf{f}_t$. Since dividing the criterion function by T does not change the solution to our minimization problem, we can think of $\widehat{\boldsymbol{\psi}}$, as the solution to the sample analog of the following set of moment conditions:

$$E\left[2\sum_{n=1}^{N}\sum_{j=1}^{J+1}\frac{\partial u_{j,t}^{(n)}}{\partial \psi_i}u_{j,t}^{(n)}\right] = E\left[s_{it}^{(1)}\right] = 0 \qquad \forall i$$

Then, we can use standard GMM asymptotic theory to obtain standard errors for $\widehat{\boldsymbol{\psi}}$:

$$\sqrt{T}(\widehat{\boldsymbol{\psi}} - \boldsymbol{\psi}) \rightarrow N\left[\mathbf{0}, \left(\mathbf{D}_{11}'\mathbf{S}_{11}^{-1}\mathbf{D}_{11}\right)^{-1}\right]$$

where $\mathbf{D}_{11} = E\left[\partial\mathbf{s}_t^{(1)}/\partial\boldsymbol{\psi}'\right]$ and $\mathbf{S}_{11} = \sum_{j=-\infty}^{\infty} E\left[\mathbf{s}_t^{(1)}\mathbf{s}_{t-j}^{(1)\prime}\right]$. Moreover, we can use the results in the previous appendix to show that, under the assumption that $u_{j,t}^{(n)}$ is orthogonal to the bond pricing factors, $\mathbf{D}_{11}$ only depends on the first derivatives of the bond price coefficients $a_j^{(n)} = -A_j^{(n)}/n$ and $\mathbf{b}_j^{(n)} = -\mathbf{B}_j^{(n)}/n$, which greatly simplifies obtaining an algebraic expression for the estimate of $\mathbf{D}_{11}$.

Stage 2. The parameters driving the price of bond and foreign exchange risks are, on the other hand, obtained from OLS regressions on the bond pricing factors:

$$\begin{aligned}
\mathbf{f}_{t+1} - \left(\widehat{\boldsymbol{\theta}}_1^Q + \widehat{\boldsymbol{\Phi}}_{11}^Q\mathbf{f}_t\right) &= \boldsymbol{\lambda}_{10} + \boldsymbol{\lambda}_{11}\mathbf{f}_t + \boldsymbol{\lambda}_{12}\mathbf{m}_t + \boldsymbol{\lambda}_{13}\Delta\mathbf{s}_t + \mathbf{v}_{1,t+1} \\
\Delta\mathbf{s}_{t+1} - \left(\widehat{\boldsymbol{\theta}}_3^Q + \widehat{\boldsymbol{\Phi}}_{31}^Q\mathbf{f}_t\right) &= \boldsymbol{\lambda}_{30} + \boldsymbol{\lambda}_{31}\mathbf{f}_t + \boldsymbol{\lambda}_{32}\mathbf{m}_t + \boldsymbol{\lambda}_{33}\Delta\mathbf{s}_t + \mathbf{v}_{3,t+1}
\end{aligned}$$

where $\widehat{\boldsymbol{\theta}}_1^Q$, $\widehat{\boldsymbol{\theta}}_3^Q$, $\widehat{\boldsymbol{\Phi}}_{11}^Q$ and $\widehat{\boldsymbol{\Phi}}_{31}^Q$ are estimates of the parameters under the risk-neutral measure obtained in the first stage.[29]

Thus, we could potentially obtain standard errors once we recast our estimation within the GMM framework using the moment conditions that are implicit in the OLS estimation:

$$E\begin{pmatrix}\mathbf{v}_{1,t+1}\otimes\mathbf{x}_t \\ \mathbf{v}_{3,t+1}\otimes\mathbf{x}_t\end{pmatrix} = E\left[\mathbf{s}_t^{(2)}\right] = 0$$

[29] To keep notation simple, we focus here on the case of unrestricted prices of risk.

where $\mathbf{x}_t = (\mathbf{f}_t', \mathbf{m}_t', \Delta\mathbf{s}_t')'$. However, inference will not be valid in this context because it doesn't take into account that $\mathbf{s}_t^{(2)}$ depends not only on the coefficients driving the price of risk, $\boldsymbol{\lambda}$, but also on the parameters governing the risk-neutral distribution, $\boldsymbol{\psi}$. In order to correct for this "generated regressors problem," we simply stack the moment conditions corresponding to both stages:

$$E\left[\begin{array}{c} \mathbf{s}_t^{(1)}(\boldsymbol{\psi}) \\ \mathbf{s}_t^{(2)}(\boldsymbol{\psi}, \boldsymbol{\lambda}) \end{array}\right] = E\left[\mathbf{s}_t(\boldsymbol{\psi}, \boldsymbol{\lambda})\right] = 0$$

and then use standard asymptotic theory to obtain

$$\sqrt{T}(\left[\left(\begin{array}{c} \widehat{\boldsymbol{\psi}} \\ \widehat{\boldsymbol{\lambda}} \end{array}\right) - \left(\begin{array}{c} \boldsymbol{\psi} \\ \boldsymbol{\lambda} \end{array}\right)\right] \rightarrow N\left[\mathbf{0}, \left(\mathbf{D}'\mathbf{S}^{-1}\mathbf{D}\right)^{-1}\right]$$

where

$$\mathbf{D} = E\left[\frac{\partial \mathbf{s}_t}{\partial(\boldsymbol{\psi}', \boldsymbol{\lambda}')'}\right] = E\left[\begin{array}{cc} \partial \mathbf{s}_t^{(1)}/\partial\boldsymbol{\psi}' & \mathbf{0} \\ \partial \mathbf{s}_t^{(2)}/\partial\boldsymbol{\psi}' & \partial \mathbf{s}_t^{(2)}/\partial\boldsymbol{\lambda}' \end{array}\right] = \left[\begin{array}{cc} \mathbf{D}_{11} & \mathbf{0} \\ \mathbf{D}_{21} & \mathbf{D}_{22} \end{array}\right]$$

and $\mathbf{S} = \sum_{j=-\infty}^{\infty} E\left[\mathbf{s}_t \mathbf{s}_{t-j}{}'\right]$

H Domestic Asset Pricing

In this appendix, we focus on the set of individual models for each country where (i) we allow for non-zero prices of risk for each country's two domestic principal components, and (ii) time-variation in the prices of risk is only driven by domestic factors. In particular, we show how to cast this collection of domestic princing models into our multi-country framework in order to compare the implied Sharpe ratios of domestic and international asset pricing models.

First note that we can represent the model for each country in terms of our canonical representation in Lemma 1 under appropriate zero restrictions on $\boldsymbol{\Gamma}^{(1)}$. In particular, we have the matrix of short-rate factor loadings:

$$\left(\begin{array}{c} r_{\$,t} \\ r_{1,t} \\ \vdots \\ r_{J,t} \end{array}\right) = \left(\begin{array}{cccc} \mathbf{1}_L' & 0 & \cdots & 0 \\ 0 & \mathbf{1}_L' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{1}_L' \end{array}\right) \left(\begin{array}{c} \widetilde{\mathbf{z}}_{\$,t} \\ \widetilde{\mathbf{z}}_{1,t} \\ \vdots \\ \widetilde{\mathbf{z}}_{J,t} \end{array}\right) \tag{H.1}$$

$$\mathbf{r}_t = \boldsymbol{\Gamma}^{(1)} \widetilde{\mathbf{z}}_t$$

where L is the number of domestic bond factors per country, $\widetilde{\mathbf{z}}_{j,t}$ is a vector that collects the set of domestic bond factors for country j, and $\mathbf{1}_L$ is a L-dimensional vector of ones.

On the other hand, the dynamics of the domestic latent factors are given by

$$\left(\begin{array}{c} \widetilde{z}_{j1,t} \\ \widetilde{z}_{j2,t} \\ \vdots \\ \widetilde{z}_{jL,t} \end{array}\right) = \left(\begin{array}{c} k_{j,\infty}^Q \\ 0 \\ \vdots \\ 0 \end{array}\right) + \left(\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & \psi_{11,j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_{11,j}^{L-1} \end{array}\right) \left(\begin{array}{c} \widetilde{z}_{j1,t-1} \\ \widetilde{z}_{j2,t-1} \\ \vdots \\ \widetilde{z}_{jL,t-1} \end{array}\right) + \left(\begin{array}{c} u_{j1,t}^Q \\ u_{j2,t}^Q \\ \vdots \\ u_{jL,t}^Q \end{array}\right) \tag{H.2}$$

$$\widetilde{\mathbf{z}}_{j,t} = \mathbf{e}_1 k_{j,\infty}^Q + \boldsymbol{\Psi}_{11,j}^Q \mathbf{z}_{j,t-1} + \mathbf{u}_{j1,t}^Q$$

In particular, we have assumed that $\widetilde{\mathbf{z}}_{j,t}$ follows an autonomous VAR(1) process under the risk-neutral measure and that the eigenvalues of $\boldsymbol{\Psi}^{Q}_{11,j}$ are distributed according to a power law relation. Again, in order to replicate a domestic level factor for each one of the countries, we set the largest eigenvalue of $\boldsymbol{\Psi}^{Q}_{11,j}$ to 1.00.

The joint dynamics of $\widetilde{\mathbf{z}}_t = (\widetilde{\mathbf{z}}_{1,t}, \widetilde{\mathbf{z}}_{2,t}, ..., \widetilde{\mathbf{z}}_{J,t})'$ can thus be cast in terms of the canonical representation in Lemma 1 in the main text with appropriate restrictions.

Then, we choose domestic state variables that are linear combinations of the domestic yields only:

$$\widetilde{\mathbf{f}}_{j,t} = \widetilde{\mathbf{P}}_j \mathbf{y}_{j,t} \tag{H.3}$$

where $\mathbf{y}_{j,t} = (y^{(1)}_{j,t}, \ldots, y^{(N)}_{j,t})'$ is a vector that collects all the yields for a given country and $\widetilde{\mathbf{P}}_j$ is a full-rank matrix of weights. We choose the first $L = 2$ principal components cross-section of yields for a given country. Stacking the domestic factors for each country, we find that, under domestic asset pricing, we also have bond factors that are linear combinations of the yields themselves.

$$\begin{pmatrix} \widetilde{\mathbf{f}}_{\$,t} \\ \widetilde{\mathbf{f}}_{1,t} \\ \vdots \\ \widetilde{\mathbf{f}}_{J,t} \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{P}}_{\$} & 0 & \cdots & 0 \\ 0 & \widetilde{\mathbf{P}}_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{\mathbf{P}}_J \end{pmatrix} \begin{pmatrix} \mathbf{y}_{1,t} \\ \mathbf{y}_{2,t} \\ \vdots \\ \mathbf{y}_{J,t} \end{pmatrix}$$

$$\widetilde{\mathbf{f}}_t = \widetilde{\mathbf{P}} \mathbf{y}_t$$

Therefore, we can use the results in Proposition 2 in the main text to obtain the self-consistency restrictions implied by the domestic pricing.

Finally, we assume the prices of risk are affine and that time-variation in the prices of risk is only driven by domestic factors, that is, domestic bond factors and domestic macroeconomic variables:

$$\boldsymbol{\lambda}_{j,t} = \boldsymbol{\lambda}_{j0} + \boldsymbol{\lambda}_{j1} \widetilde{\mathbf{f}}_{j,t} + \boldsymbol{\lambda}_{j2} \mathbf{m}_{j,t} \tag{H.4}$$

where $\mathbf{m}_{j,t} = (g_{j,t}, \pi_{j,t})'$ is a vector that collects country j's growth and inflation rates. We can stack (H.4) for each country to obtain that

$$\begin{pmatrix} \boldsymbol{\lambda}_{\$,t} \\ \boldsymbol{\lambda}_{1,t} \\ \vdots \\ \boldsymbol{\lambda}_{J,t} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\lambda}_{\$0} \\ \boldsymbol{\lambda}_{10} \\ \vdots \\ \boldsymbol{\lambda}_{J0} \end{pmatrix} + \begin{pmatrix} \boldsymbol{\lambda}_{\$1} & 0 & \cdots & 0 \\ 0 & \boldsymbol{\lambda}_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\lambda}_{J1} \end{pmatrix} \begin{pmatrix} \widetilde{\mathbf{f}}_{\$,t} \\ \widetilde{\mathbf{f}}_{1,t} \\ \vdots \\ \widetilde{\mathbf{f}}_{J,t} \end{pmatrix}$$

$$+ \begin{pmatrix} \boldsymbol{\lambda}_{\$2} & 0 & \cdots & 0 \\ 0 & \boldsymbol{\lambda}_{12} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\lambda}_{J2} \end{pmatrix} \begin{pmatrix} \mathbf{m}_{\$,t} \\ \mathbf{m}_{1,t} \\ \vdots \\ \mathbf{m}_{J,t} \end{pmatrix}$$

One can thus understand the domestic asset pricing model as a multi-country model where zero restrictions are imposed on the general characterization of the prices of risk $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\lambda} \mathbf{x}_t$.

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Table 1
Summary Statistics

	Mean	Std. Dev.	Skewness	Excess Kurtosis	Autocorrelation 1	12
U.S.						
1-year yield	6.213	3.224	0.540	0.272	0.990	0.875
2-year yield	6.462	3.115	0.487	0.079	0.992	0.893
5-year yield	6.955	2.839	0.547	-0.082	0.993	0.914
10-year yield	7.332	2.535	0.675	-0.131	0.993	0.905
Inflation	4.203	2.817	1.351	1.769	0.989	0.744
Growth	1.912	4.695	-1.089	1.553	0.974	0.095
U.K.						
Rate of Depreciation	-0.999	11.878	-0.458	-0.200	0.930	0.007
1-year yield	7.918	3.324	0.057	-0.848	0.992	0.881
2-year yield	8.058	3.216	0.013	-0.962	0.992	0.906
5-year yield	8.332	3.165	0.039	-1.145	0.993	0.928
10-year yield	8.544	3.267	0.072	-1.306	0.992	0.946
Inflation	5.860	5.071	1.578	2.007	0.975	0.673
Growth	0.546	4.205	-1.093	1.754	0.900	0.036
Germany						
Rate of Depreciation	1.664	12.368	0.109	-0.309	0.931	0.116
1-year yield	5.168	2.394	0.745	0.253	0.994	0.807
2-year yield	5.424	2.258	0.524	-0.226	0.994	0.831
5-year yield	6.010	2.026	0.196	-0.766	0.993	0.858
10-year yield	6.454	1.774	-0.042	-1.031	0.990	0.867
Inflation	2.568	1.697	0.564	-0.364	0.979	0.674
Growth	1.093	5.573	-2.014	6.377	0.919	-0.015
Canada						
Rate of Depreciation	-0.321	6.592	-0.036	1.508	0.934	0.066
1-year yield	7.024	3.543	0.484	-0.034	0.992	0.881
2-year yield	7.196	3.344	0.466	-0.024	0.991	0.895
5-year yield	7.595	3.064	0.363	-0.342	0.992	0.926
10-year yield	8.004	2.964	0.378	-0.339	0.994	0.937
Inflation	4.218	3.173	0.864	-0.328	0.988	0.816
Growth	2.185	4.644	-0.672	1.837	0.937	-0.002

Note: Data are sampled monthly from January 1975 to December 2009. All variables are measured in percentage points per year.

Table 2
Principal Components Analysis

Panel A: Per cent variation in yield curves explained by the first k domestic PCs

k	U.S.	U.K.	GER	CAN
1	98.44	98.03	96.61	98.61
2	99.90	99.86	99.85	99.89
3	99.97	99.98	99.98	99.96
4	99.98	100.00	100.00	99.99
5	99.99	100.00	100.00	100.00

Panel B: Per cent variation in yield curves explained by the first k global PCs

k	per cent	k	per cent
1	91.48	6	99.68
2	95.44	7	99.85
3	97.53	8	99.90
4	98.58	9	99.94
5	99.38	10	99.95

Panel C: RMSE (in basis points) of a regression of yields on the first k PCs

k	U.S.	U.K.	GER	CAN	Global
Domestic PCs					
1	35.60	45.04	37.45	36.95	38.93
2	9.20	11.81	7.81	10.61	9.97
3	5.04	4.14	2.89	6.54	4.84
Global PCs					
1	83.89	95.03	86.16	64.05	83.06
2	47.95	59.57	77.71	53.77	60.78
3	44.95	44.86	35.18	52.38	44.76
4	38.13	25.23	27.38	41.86	33.88
5	23.28	25.05	22.48	18.54	22.46
6	11.98	15.94	21.44	13.26	16.07
7	10.92	11.34	7.86	13.13	10.98
8	9.26	8.39	7.54	10.63	9.02
9	8.00	4.82	7.13	8.32	7.20
10	5.82	4.23	6.06	7.96	6.16
11	5.65	3.63	4.10	6.35	5.06
12	4.56	3.13	2.65	5.57	4.14
13	4.41	3.11	2.40	3.46	3.42
14	4.25	3.08	2.33	1.80	3.01
15	3.95	1.64	2.26	1.69	2.56

Note: Data are sampled monthly from January 1975 to December 2009.

Table 3
Unspanned Risks

LHS\RHS	PC1-PC8	PC1-PC16	PC1-PC8, all macro	PC1-PC16, all macro
Global Growth	22.26	28.67	-	-
Global Inflation	75.95	78.85	-	-
Growth U.S.	19.42	25.39	-	-
Growth U.K.	17.35	31.91	-	-
Growth Germany	38.60	45.10	-	-
Growth Canada	21.20	36.51	-	-
Inflation U.S.	71.23	75.12	-	-
Inflation U.K.	75.33	80.81	-	-
Inflation Germany	79.76	82.85	-	-
Inflation Canada	78.11	81.60	-	-
USD/GBP Rate of Depreciation	21.63	29.40	43.82	46.39
USD/EUR Rate of Depreciation	41.22	49.05	55.84	61.59
USD/CAD Rate of Depreciation	17.30	22.78	36.47	40.90

Note: R^2s (in per cent) from contemporaneous regression of LHS variables on RHS variables.

Table 4
Risk Neutral Parameter Estimates

	$k^{Q}_{j,\infty}$	$\mathbf{\Gamma}^{(1)}$ 1	2	3	4	5	6	7	8
U.S.	−0.0010	0.6511	0.7530	−1.2112	−1.7424	1.9096	0.6838	0.6452	−0.6888
	(0.0004)	(0.0151)	(0.0164)	(0.0430)	(0.0801)	(0.0779)	(0.0267)	(0.0315)	(0.0662)
	[0.0111]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]
U.K.	0.0166	1.7755	0.2755	1.2347	1.8161	−0.9545	−0.0513	−0.2230	2.0300
	(0.0007)	(0.0063)	(0.0196)	(0.0484)	(0.0841)	(0.0878)	(0.0385)	(0.0543)	(0.1194)
	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[0.1831]	[< 0.001]	[< 0.001]
Germany	0.0258	−1.4297	−0.5078	1.0881	0.8998	−0.2303	0.1080	0.3561	−0.9437
	(0.0015)	(0.0110)	(0.0243)	(0.0491)	(0.0584)	(0.0540)	(0.0210)	(0.0332)	(0.0815)
	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]
Canada	−0.0181	0.0030	0.4793	−0.1116	0.0264	0.2752	0.2594	0.2217	0.6025
	(0.0009)	(0.0131)	(0.0077)	(0.0189)	(0.0244)	(0.0224)	(0.0073)	(0.0097)	(0.0260)
	[< 0.001]	[0.8180]	[< 0.001]	[< 0.001]	[0.2792]	[< 0.001]	[< 0.001]	[< 0.001]	[< 0.001]

Note: Estimates of the long-run means of the short-rates under the risk neutral measure and the factor loadings of the short-term interest rate in the canonical model. Newey-West standard errors are given in parentheses, and *p*-values in square brackets. The parameters driving the power law relation for the eigenvalues of $\mathbf{\Psi}^{Q}_{11}$ in equation (22) are set to $\overline{\psi}^{Q}_{11} = 1.00$ and $\varphi = 0.94$.

Table 5
Model Fit in Basis Points

	RMSPE			MAPE		
	Affine	OLS	Difference	Affine	OLS	Difference
U.S.	9.84	9.26	0.59	6.78	6.57	0.22
U.K.	9.29	8.39	0.90	6.50	5.76	0.74
Germany	7.78	7.54	0.24	5.47	5.31	0.15
Canada	11.27	10.63	0.64	8.23	7.88	0.35
Global	9.63	9.02	0.60	6.75	6.38	0.37

Note: Affine model fit in basis points ($1 = 0.01$ per cent). RMSPE gives the root mean squared pricing error, and MAPE gives mean absolute pricing error. "Affine" provides the fit of the multi-country term structure model, while "OLS" provides the the model fit of a regression of yields on the first eight global principal components. "Difference" provides the loss of fit in basis points of estimating an affine term structure model instead of unrestricted OLS regressions.

Table 6
Price of risk estimates

Panel A: Bond Risk Premia with Global Asset Pricing Restrictions

	Constant	Global Level	Global Slope	Global Growth	Global Inflation
Global Level	0.0127	−0.1662	−0.7308	0.1755	1.3580
	(0.0307)	(0.0854)	(0.4238)	(0.2199)	(0.5071)
	[0.6795]	[0.0516]	[0.0846]	[0.4249]	[0.0074]
Global Slope	0.0107	0.0077	−0.1729	−0.1523	−0.0515
	(0.0063)	(0.0155)	(0.0789)	(0.0447)	(0.0813)
	[0.0913]	[0.6223]	[0.0284]	[0.0007]	[0.5263]

Note: Estimates of the parameters governing bond expected excess returns for the model under the assumption of global asset pricing. Newey-West standard errors are given in parentheses, and p-values in square brackets.

Panel B: Foreign Exchange Risk Premia with Carry Trade Fundamentals

	Constant	Interest Rate Differential	Growth Differential	Inflation Differential
USD/GBP	−0.0214	−2.2407	−0.0247	0.9503
	(0.0216)	(1.0073)	(0.3199)	(0.5143)
	[0.3229]	[0.0261]	[0.9384]	[0.0646]
USD/EUR	0.0505	−0.4275	−0.6465	−1.9186
	(0.0200)	(0.9252)	(0.3041)	(1.0774)
	[0.0114]	[0.6441]	[0.0335]	[0.0750]
USD/CAD	−0.0071	−1.2602	−0.3139	−0.3807
	(0.0101)	(0.5584)	(0.1757)	(0.4422)
	[0.4829]	[0.0240]	[0.0740]	[0.3893]

Note: Estimates of the parameters governing foreign exchange expected excess returns for the model under the assumption of carry trade fundamentals. Newey-West standard errors are given in parentheses, and p-values in square brackets.

Table 7
Consistency of model's interest and exchange rate forecasts with survey expectations

Panel A: Unrestricted model

	$\widehat{\alpha}$	$\widehat{\beta}$	$H_0 : \alpha = 0$ $\beta = 1$	$H_0 : \boldsymbol{\gamma} = 0$	VR-model	VR-other
10-year Treasury par bond yields						
U.S.	0.477	0.977	288.2	1236.1	0.836	0.145
	(0.098)	(0.017)	[< 0.001]	[< 0.001]		
U.K.	−0.704	1.157	277.9	1262.4	0.942	0.049
	(0.063)	(0.011)	[< 0.001]	[< 0.001]		
Germany	−0.489	1.142	319.1	1125.3	0.900	0.089
	(0.059)	(0.011)	[< 0.001]	[< 0.001]		
Canada	0.232	1.009	232.8	2147.6	0.872	0.116
	(0.072)	(0.011)	[< 0.001]	[< 0.001]		
Exchange Rates						
USD/GBP	0.183	0.878	7.4	176.5	0.650	0.182
	(0.093)	(0.058)	[0.024]	[< 0.001]		
USD/EUR	0.485	0.586	200.2	544.2	0.654	0.263
	(0.034)	(0.029)	[< 0.001]	[< 0.001]		
USD/CAD	0.197	0.749	1092.3	2457.1	0.858	0.131
	(0.006)	(0.008)	[< 0.001]	[< 0.001]		

Note: The upper portion of Panel A presents results from the OLS regresions of the Consensus Economics survey forecast of the ten-year par bond yields on a constant (α), the forecast of the same yield implied by the unrestricted term structure model (β), and the set of orthogonalized (with respect to the implied forecast) bond and macroeconomic factors ($\boldsymbol{\gamma}$). The lower portion of the Panel presents the same regression results for the exchange rates. The column labeled "VR-model" shows the ratio of the variance explained by the forecast from the unrestricted term structure model to the variance of the survey forecast. The column "VR-other" shows the ratio of the variance explained by the orthogonalized bond and macroeconomic factors to the variance of the survey forecast. Newey-West standard errors are given in parentheses, and p-values in square brackets.

Table 7 (cont.)
Consistency of model's interest and exchange rate forecasts with survey expectations

Panel B: Restricted model

	$\widehat{\alpha}$	$\widehat{\beta}$	$H_0 : \alpha = 0$ $\beta = 1$	$H_0 : \boldsymbol{\gamma} = 0$	VR-model	VR-other
10-year Treasury bond yields						
U.S.	0.645	0.905	55.9	188.1	0.957	0.026
	(0.090)	(0.014)	[< 0.001]	[< 0.001]		
U.K.	0.401	0.961	132.8	372.4	0.971	0.020
	(0.051)	(0.009)	[< 0.001]	[< 0.001]		
Germany	0.153	0.964	20.4	153.6	0.976	0.013
	(0.055)	(0.010)	[< 0.001]	[< 0.001]		
Canada	0.747	0.884	118.9	331.1	0.974	0.015
	(0.071)	(0.011)	[< 0.001]	[< 0.001]		
Exchange Rates						
USD/GBP	0.219	0.860	72.5	380.9	0.917	0.052
	(0.028)	(0.017)	[< 0.001]	[< 0.001]		
USD/EUR	0.309	0.735	569.5	511.0	0.912	0.069
	(0.018)	(0.014)	[< 0.001]	[< 0.001]		
USD/CAD	0.149	0.820	713.9	307.6	0.984	0.007
	(0.007)	(0.009)	[< 0.001]	[< 0.001]		

Note: The upper portion of Panel B presents results from the OLS regresions of the Consensus Economics survey forecast of the ten-year par bond yields on a constant (α), the forecast of the same yield implied by the restricted term structure model (β), and the set of orthogonalized (with respect to the implied forecast) bond and macroeconomic factors ($\boldsymbol{\gamma}$). The lower portion of the Panel presents the same regression results for the exchange rates. The column labeled "VR-model" shows the ratio of the variance explained by the forecast from the restricted term structure model to the variance of the survey forecast. The column "VR-other" shows the ratio of the variance explained by the orthogonalized bond and macroeconomic factors to the variance of the survey forecast. Newey-West standard errors are given in parentheses, and p-values in square brackets.

Table 8
The Conundra:
May 04 - July 05

	Realized one-year yield	Realized ten-year yield	Fitted ten-year yield	Expectation component	Term Premia component	Residual
U.S.						
May-04	1.64%	4.74%	4.74%	3.84%	0.90%	0.00%
Jul-05	3.86%	4.33%	4.44%	4.25%	0.19%	−0.11%
Change (in bps)	222.00	−41.00	−30.41	41.33	−71.74	−10.59
U.K.						
May-04	4.47%	4.96%	4.94%	4.78%	0.16%	0.01%
Jul-05	4.21%	4.29%	4.26%	4.95%	−0.69%	0.03%
Change (in bps)	−25.83	−66.81	−67.95	17.00	−84.95	1.14
Germany						
May-04	2.17%	4.40%	4.42%	3.77%	0.65%	−0.03%
Jul-05	2.14%	3.26%	3.34%	3.19%	0.15%	−0.08%
Change (in bps)	−2.50	−113.80	−108.45	−57.98	−50.47	−5.35
Canada						
May-04	2.13%	4.78%	4.84%	4.45%	0.39%	−0.06%
Jul-05	2.88%	3.96%	3.83%	4.27%	−0.44%	0.13%
Change (in bps)	75.00	−82.00	−101.62	−18.04	−83.58	19.62

Note: The first column presents the observed zero-coupon one-year Treasury yields in May 2004, July 2005 and the change between the two dates. The second presents the observed zero-coupon ten-year Treasury yields while the third column presents the values implied by the restricted multi-country affine term structure model. The fourth and fifth columns present the decomposition of the ten-year yields into their expectation and term premia components: $y_{j,t}^{(n)} = \frac{1}{n}\sum_{h=1}^{n} E_t y_{j,t+h-1}^{(1)} + tp_{j,t}^{(n)}$. The final column presents the residuals (i.e. the differences between the observed ten-year yields and the ten-year yields implied by the multi-country affine term structure model).

Figure 1: Bond factor loadings: affine term structure versus OLS estimates

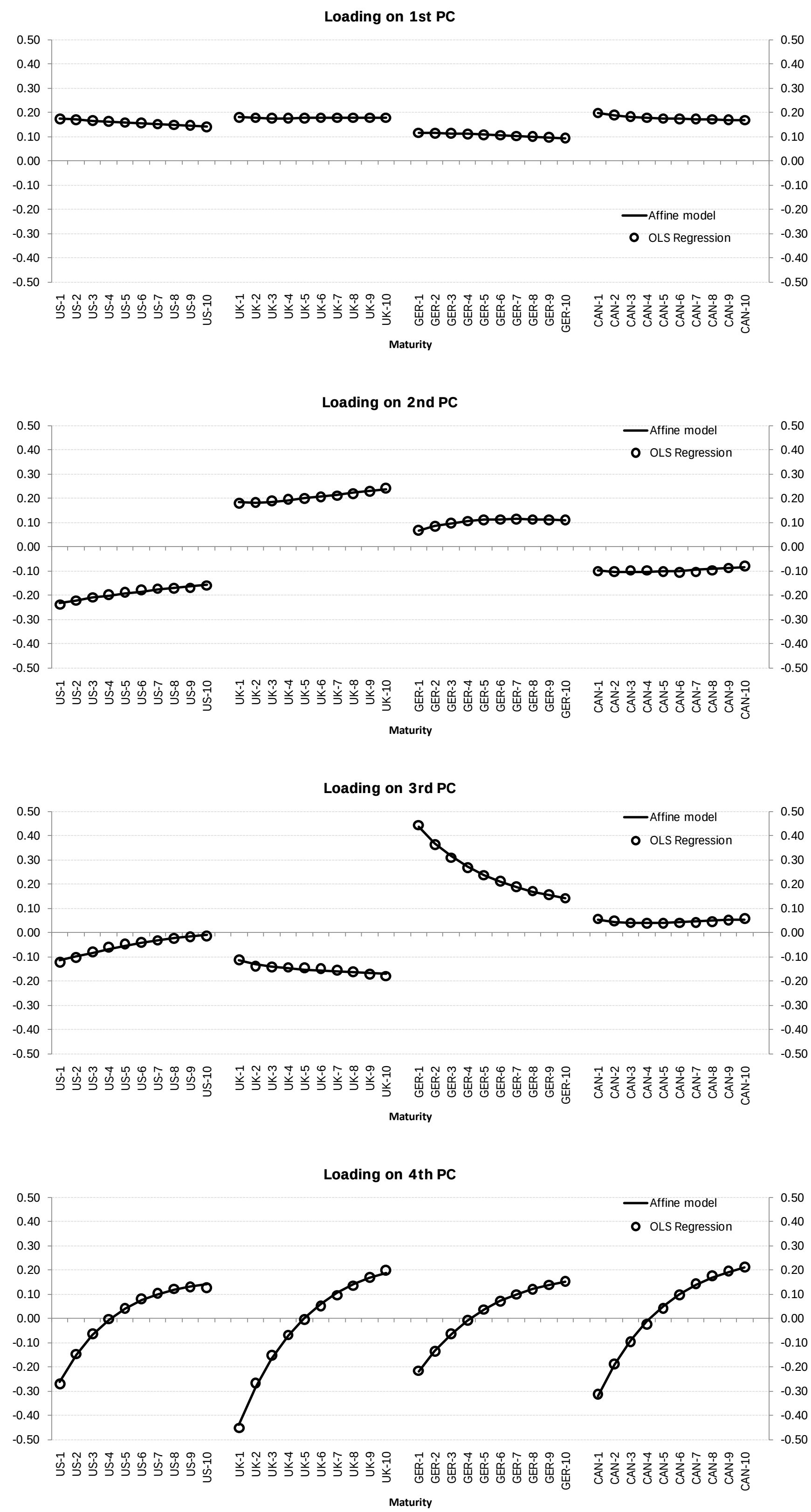
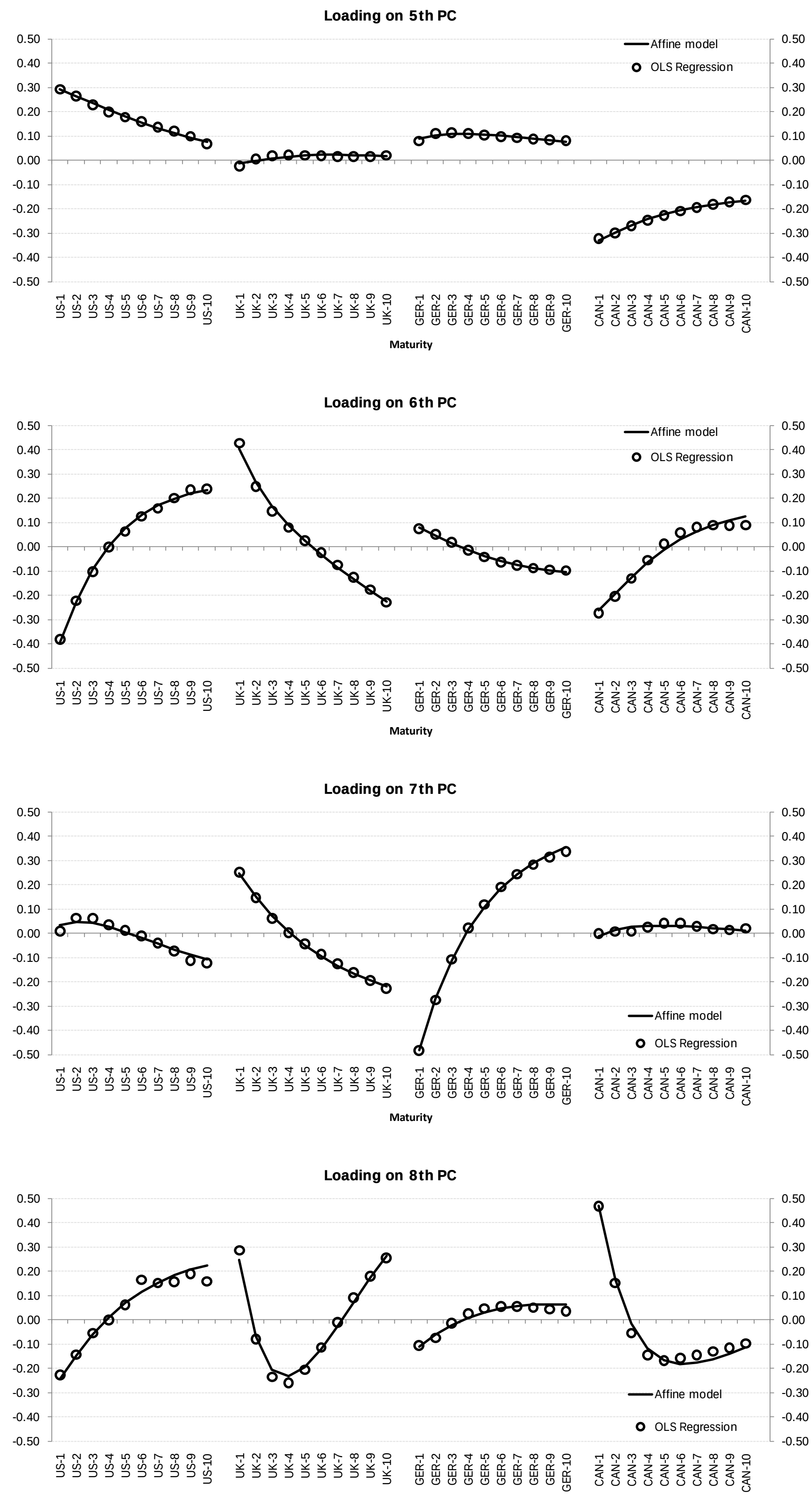
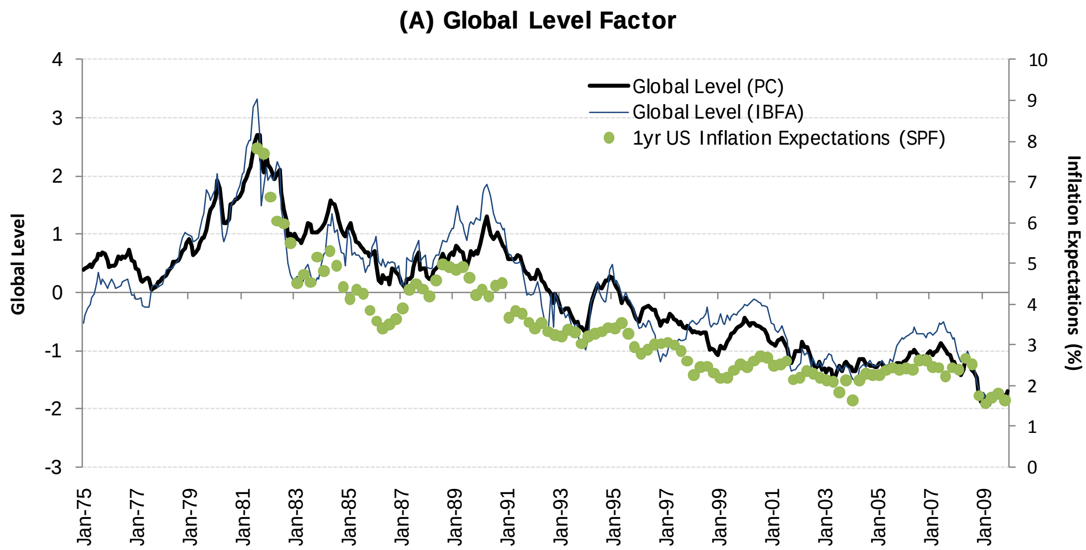


Figure 1 (cont.): Bond factor loadings: affine term structure versus OLS estimates

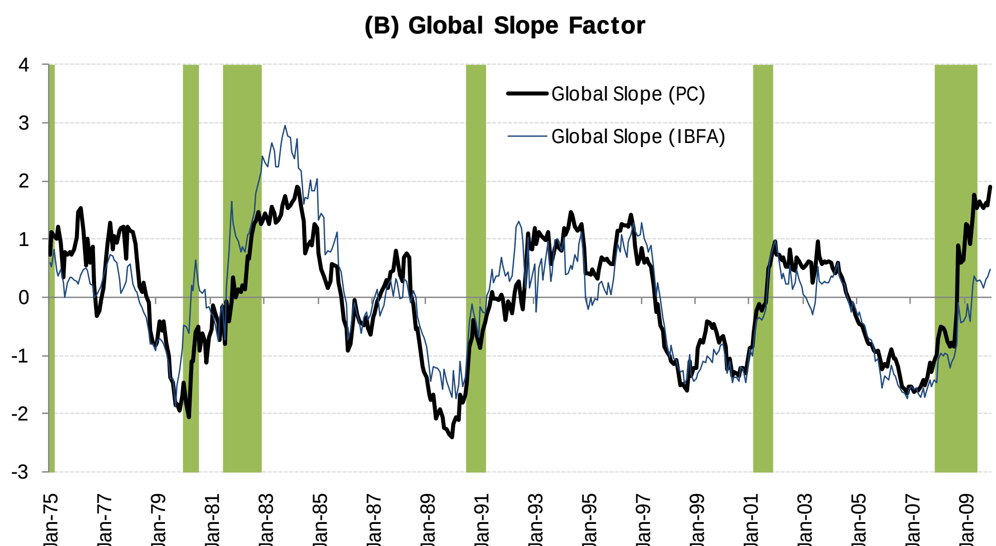


Note: Affine model bond yield loadings, $\mathbf{b}_j^{(n)}$ in $y_{j,t}^{(n)} = a_{j,t}^{(n)} + \mathbf{b}_j^{(n)\prime}\mathbf{f}_t$. The solid line gives the loadings implied by the multi-country affine term structure model. The circles give the loadings implied by the principal component decomposition of the cross-section of global yields (i.e., OLS regression coefficients of yields on the factors).

Figure 2: Principal components versus IBFA factor estimates

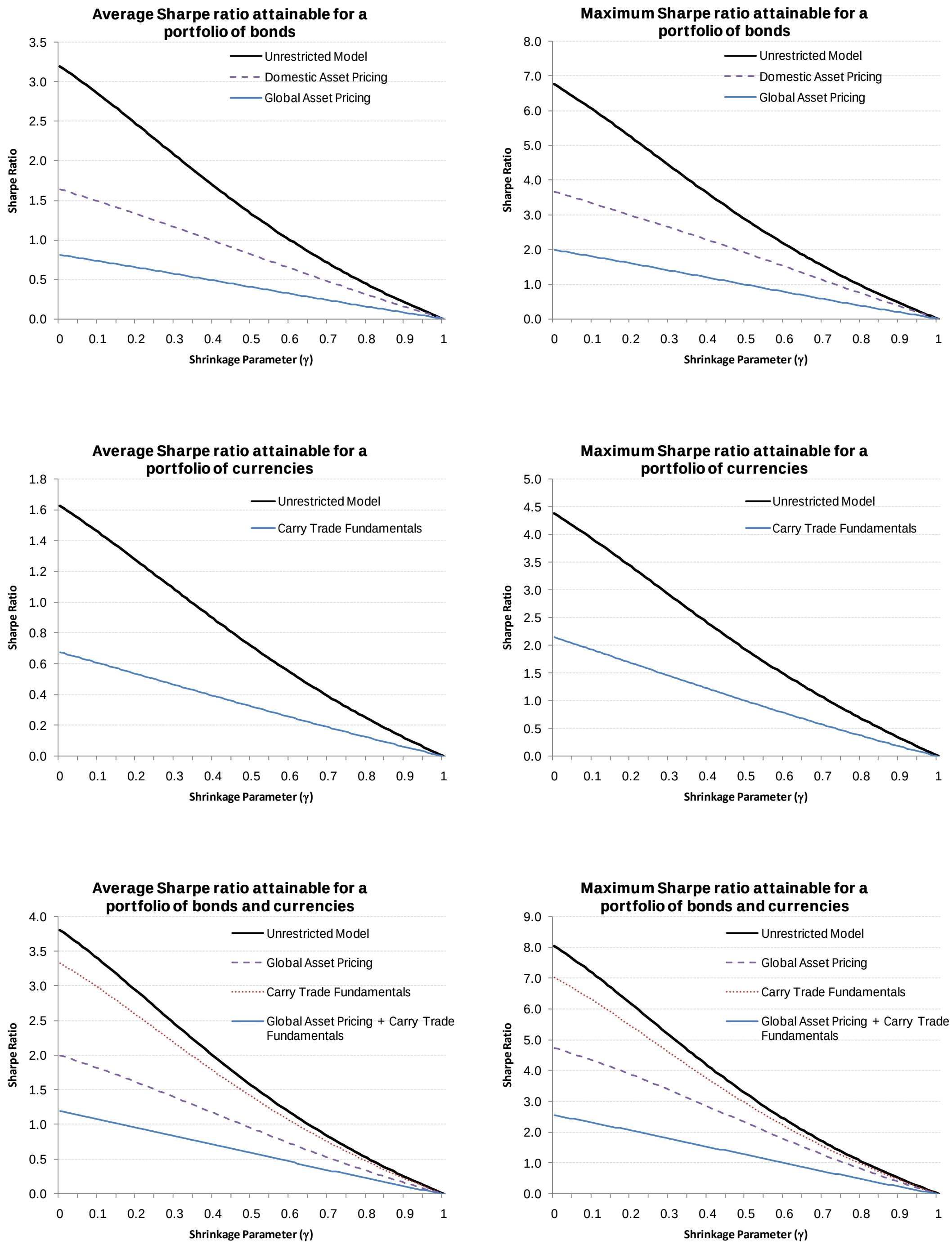


Note: The thick line gives the estimated global level factor from PCA (i.e. the first principal component), while the thin line gives the estimated global level factor from IBFA (left hand side scale). One-year U.S. inflation expectations (dots) from the Survey of Professional Forecasters (SPF) are also shown (right hand scale).



Note: The thick line gives the estimated global slope factor from PCA (i.e. the fourth principal component), while the thin line gives the estimated global slope factor from IBFA. Shaded areas indicate NBER recession dates for the U.S.

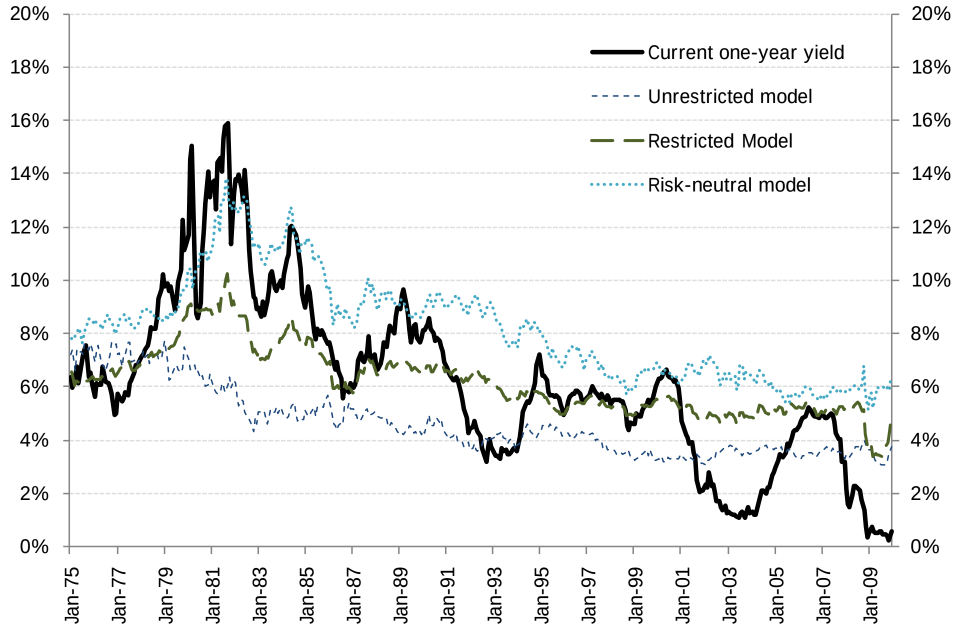
Figure 3: Sharpe ratios



Note: The left panels present the average of the time series of conditional maximum Sharpe ratios that can be attained by investing only in bonds (top panel), only in currencies (center panel) and in both bonds and currencies (lower panel) for different values of the shrinkage parameter and different restrictions on the prices of risk. The right panels present the maximum of such time series of maximal conditional Sharpe ratios.

Figure 4: Long-run expectations of one-year yields

(A) Ten-year ahead forecasts of one-year yield: U.S.



(B) Ten-year ahead forecasts of one-year yield: U.K.

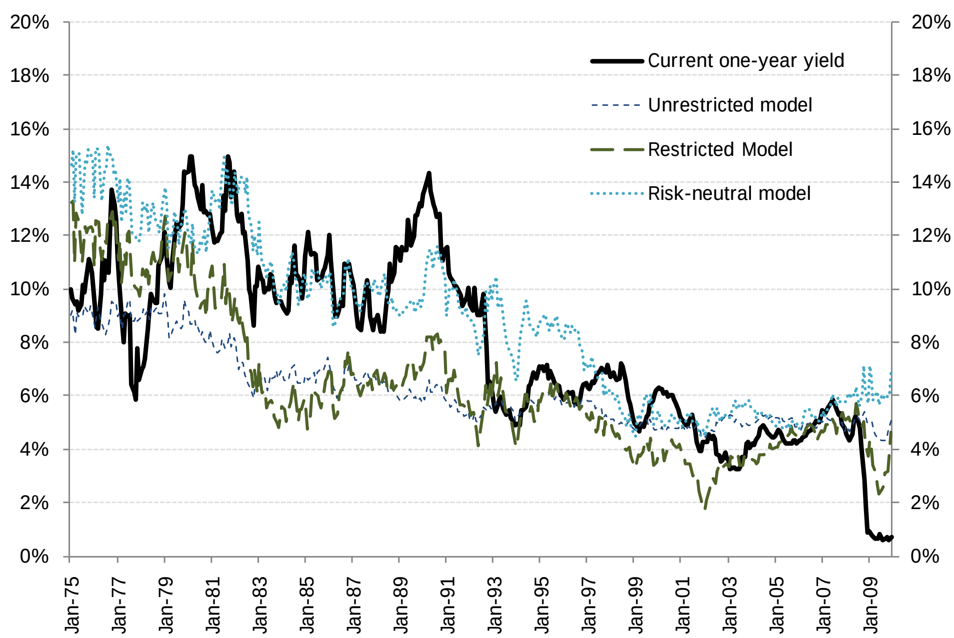
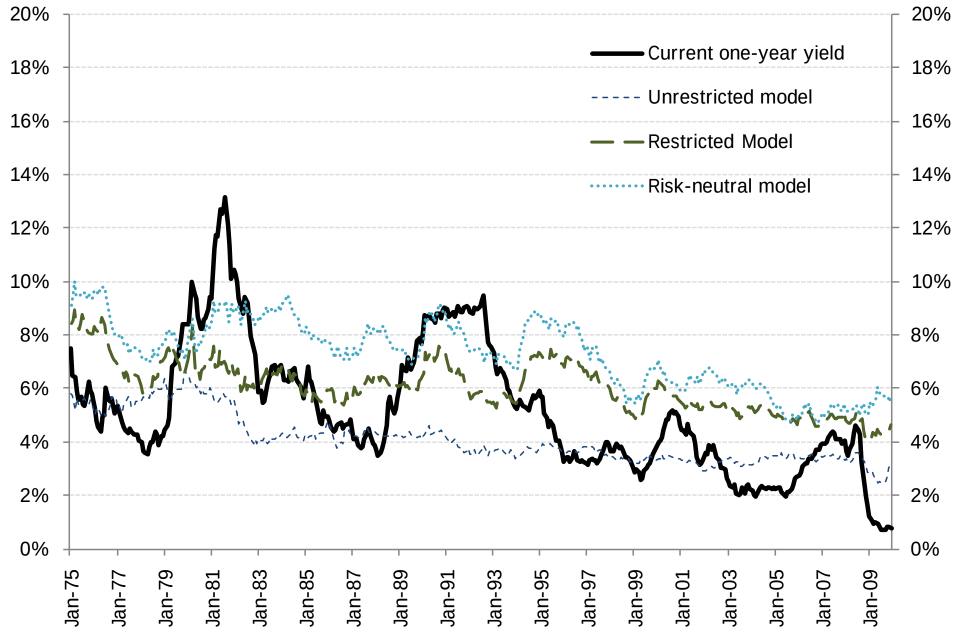
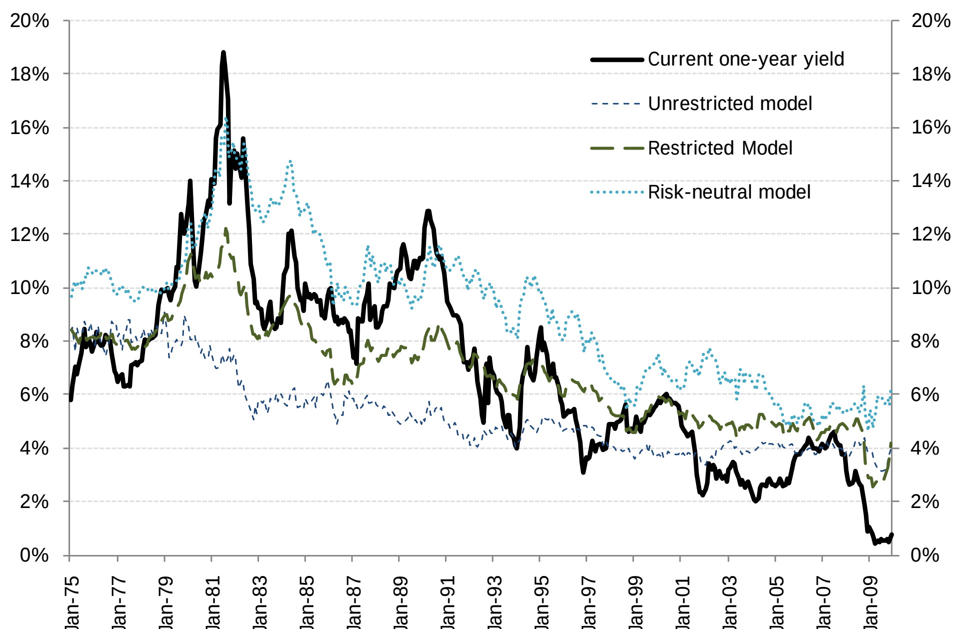


Figure 4 (cont.): Long-run expectations of one-year yields

(C) Ten-year ahead forecasts of one-year yield: Germany

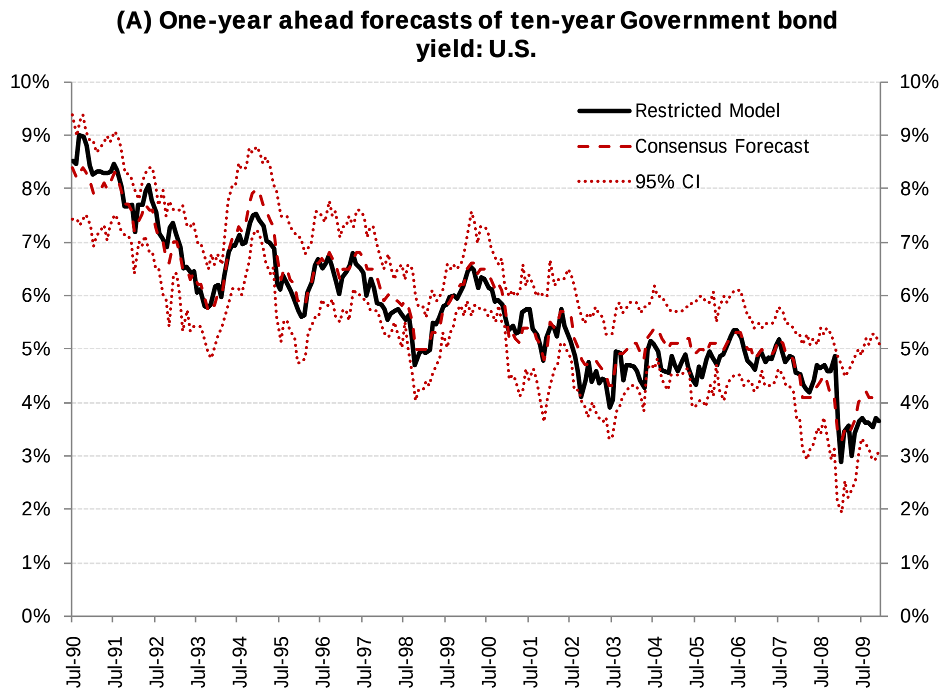


(D) Ten-year ahead forecasts of one-year yield: Canada



Note: The figures show the current one-year yield and their ten-year ahead forecasts generated by the unrestricted affine term structure model, the restricted affine term structure model (which includes the assumptions of global asset pricing, carry trade fundamentals and a shrinkage parameter equal to a 0.5), and a risk-neutral model where the prices of risk are set to zero.

Figure 5: Consistency with survey expectations of interest rates



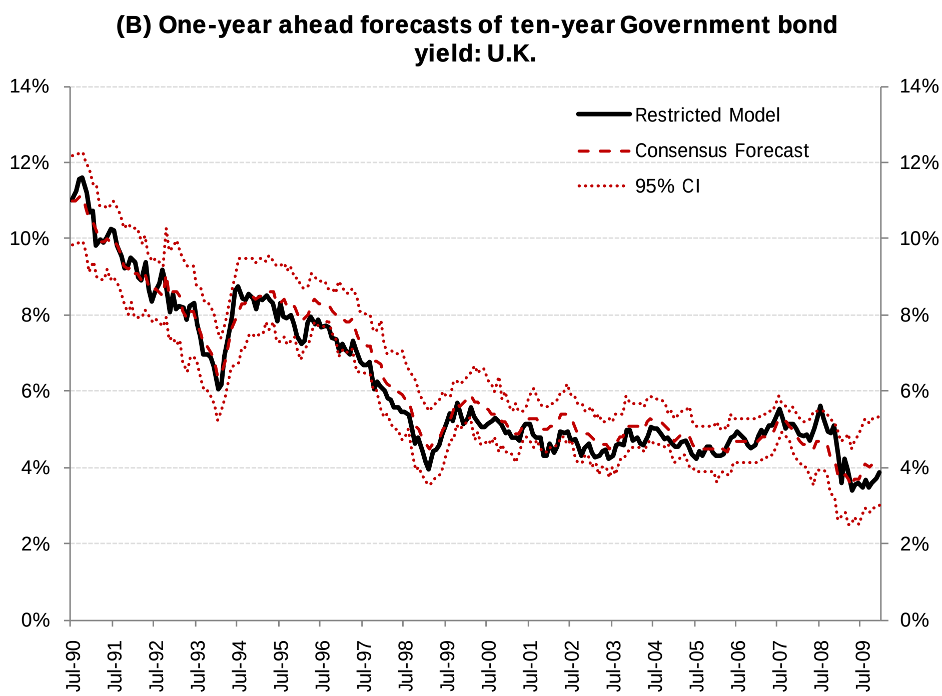
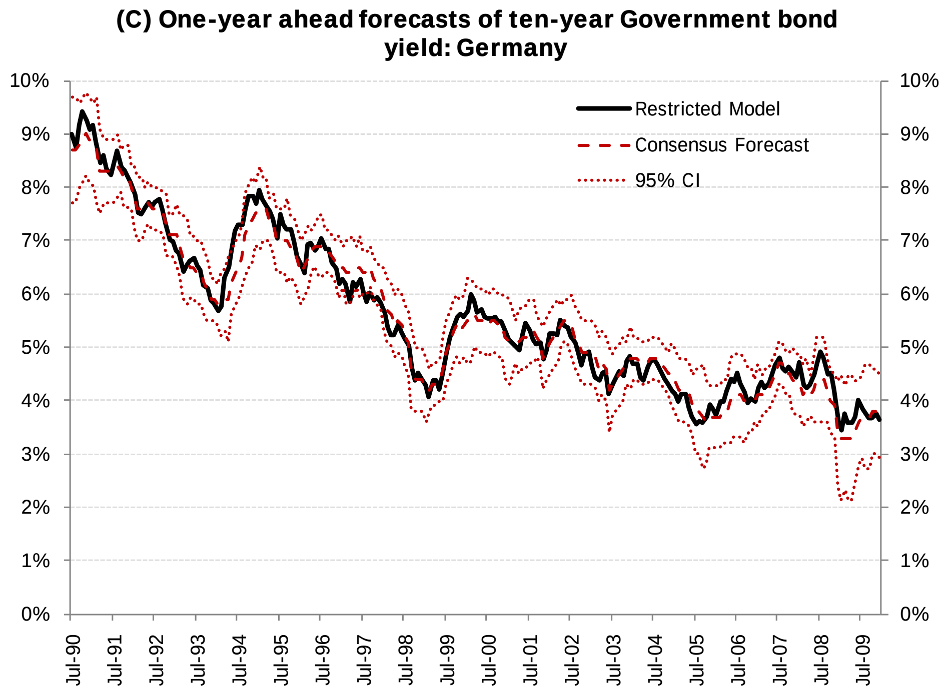
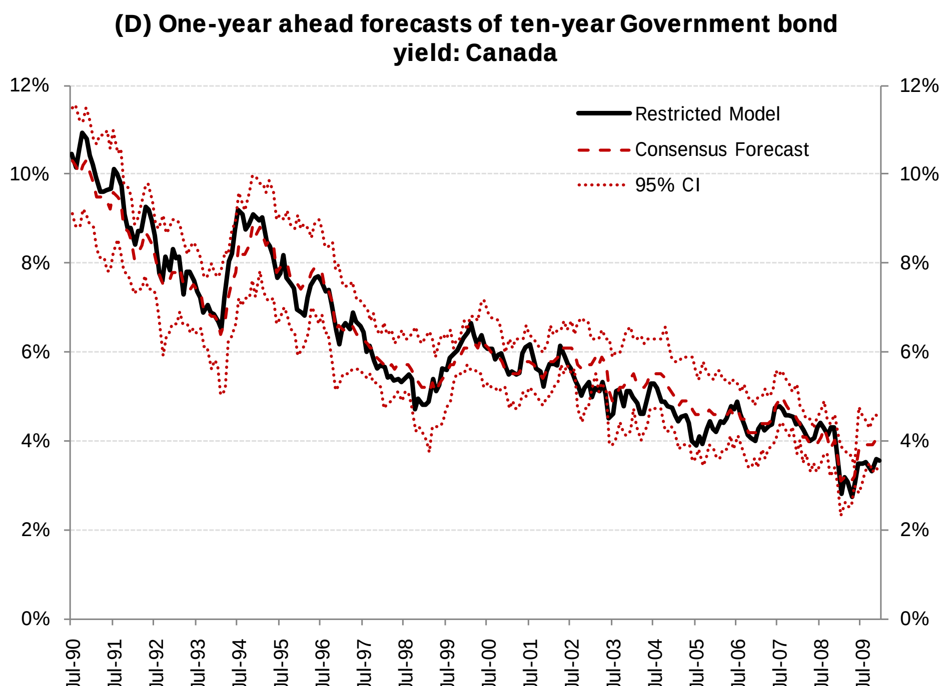


Figure 5 (cont.): Consistency with survey expectations of interest rates

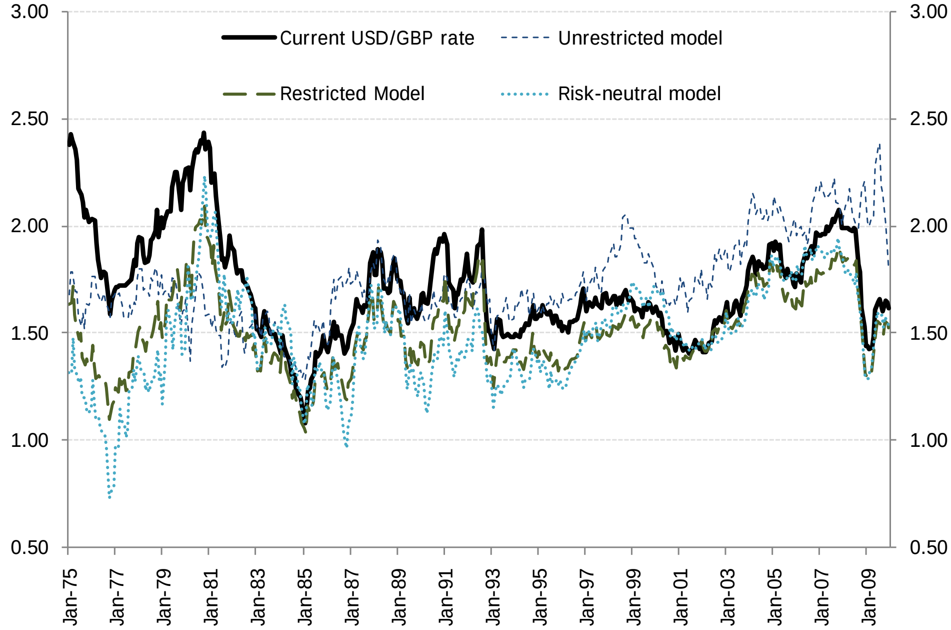




Note: The figures show the one-year ahead forecasts of ten-year par Government bond yields generated by the restricted affine term structure model (which includes the assumptions of global asset pricing, carry trade fundamentals and a shrinkage parameter equal to a 0.5), and the Consensus Economics survey forecasts (with 95% confidence intervals constructed under the assumption of normality).

Figure 6: Long-run expectations of exchange rates

(A) Ten-year ahead forecast of USD/GBP exchange rate



(B) Ten-year ahead forecast of USD/EUR exchange rate

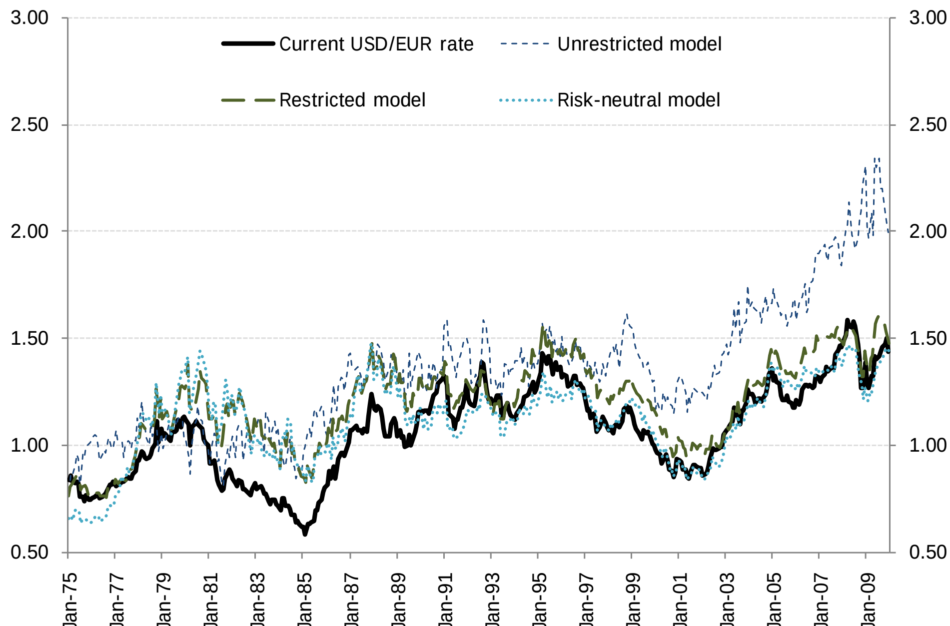
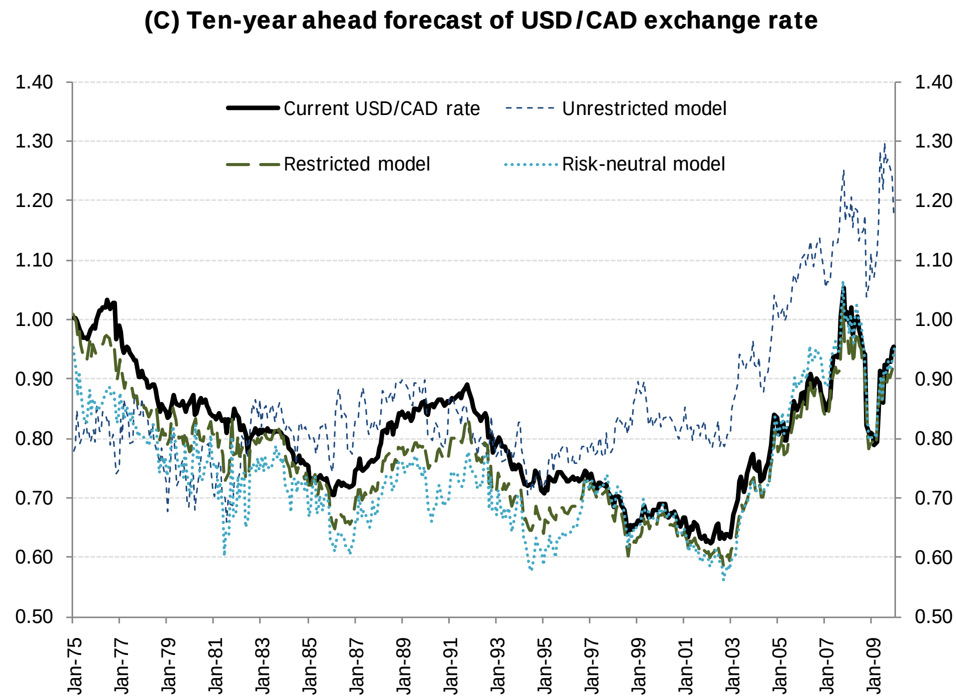


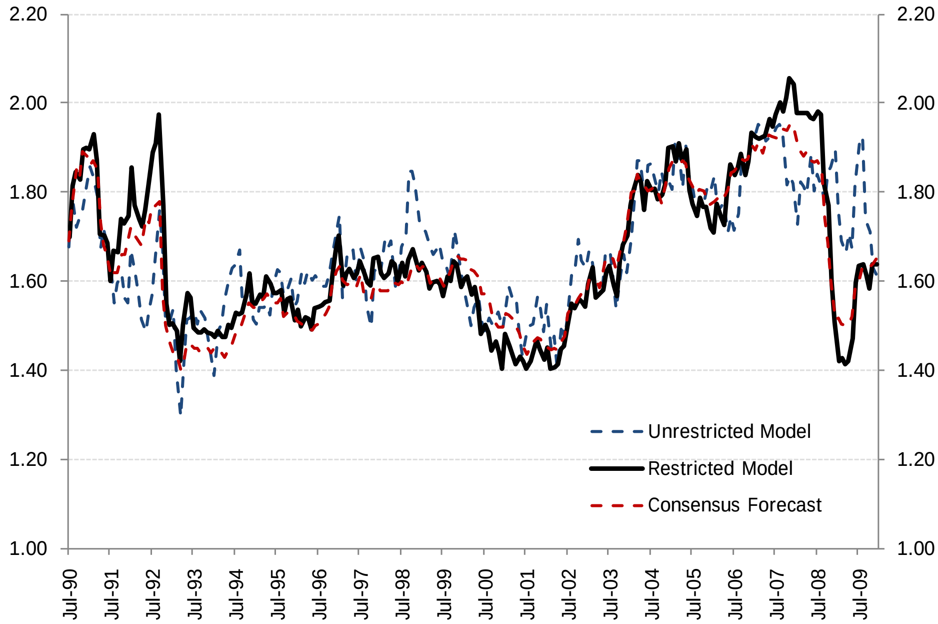
Figure 6 (cont.): Long-run expectations of exchange rates



Note: Current spot foreign exchange rates and their ten-year ahead forecasts generated by the unrestricted affine term structure model, the restricted affine term structure model (which includes the assumptions of global asset pricing, carry trade fundamentals and a shrinkage parameter equal to a 0.5), and a risk-neutral model where the prices of risk are set to zero.

Figure 7: Consistency with survey expectations of exchange rates

(A) One-year ahead forecasts of USD/GBP exchange rate



(B) One-year ahead forecasts of USD/EUR exchange rate

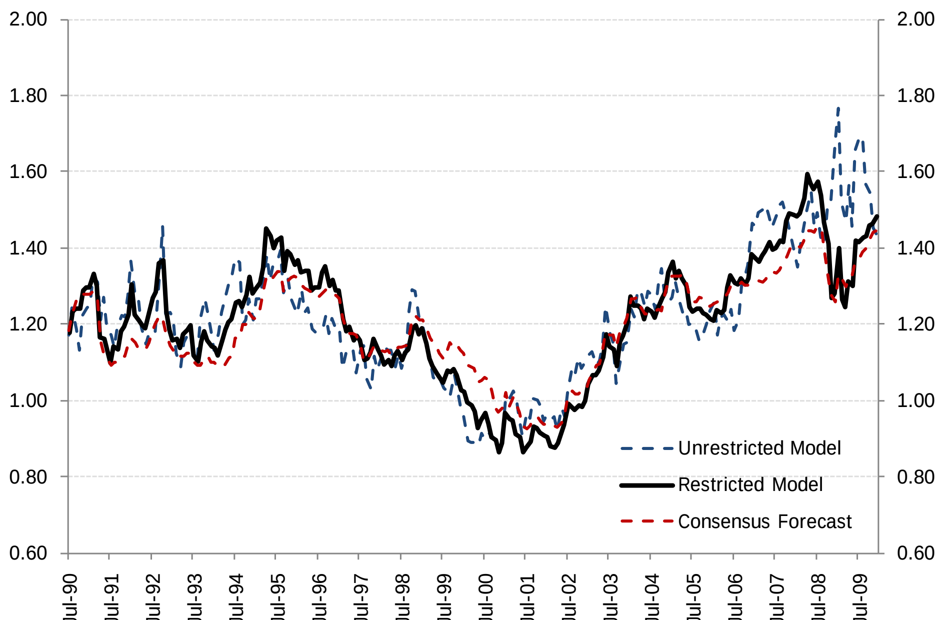
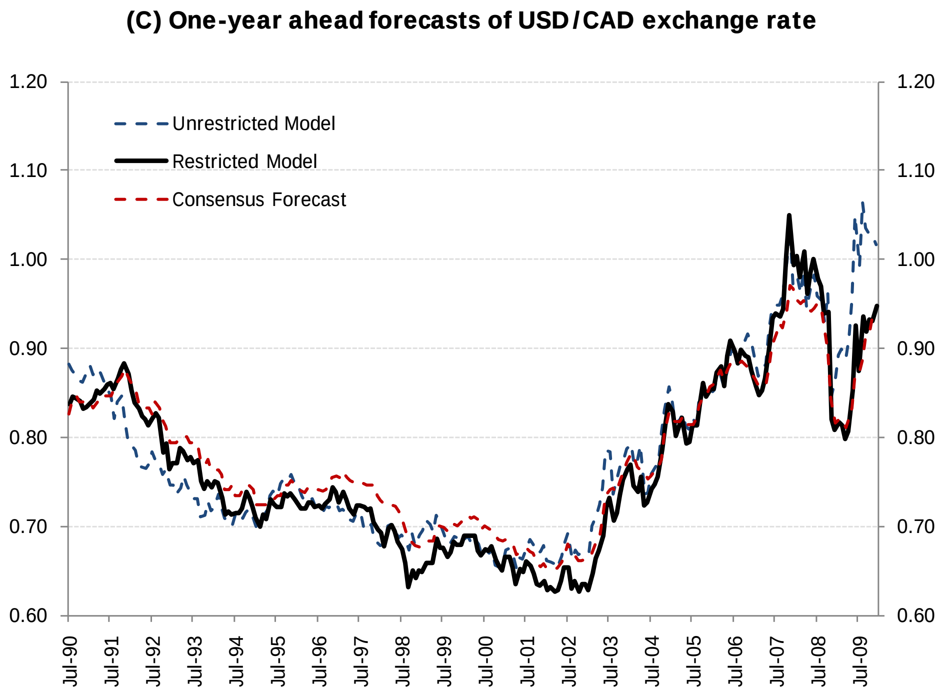


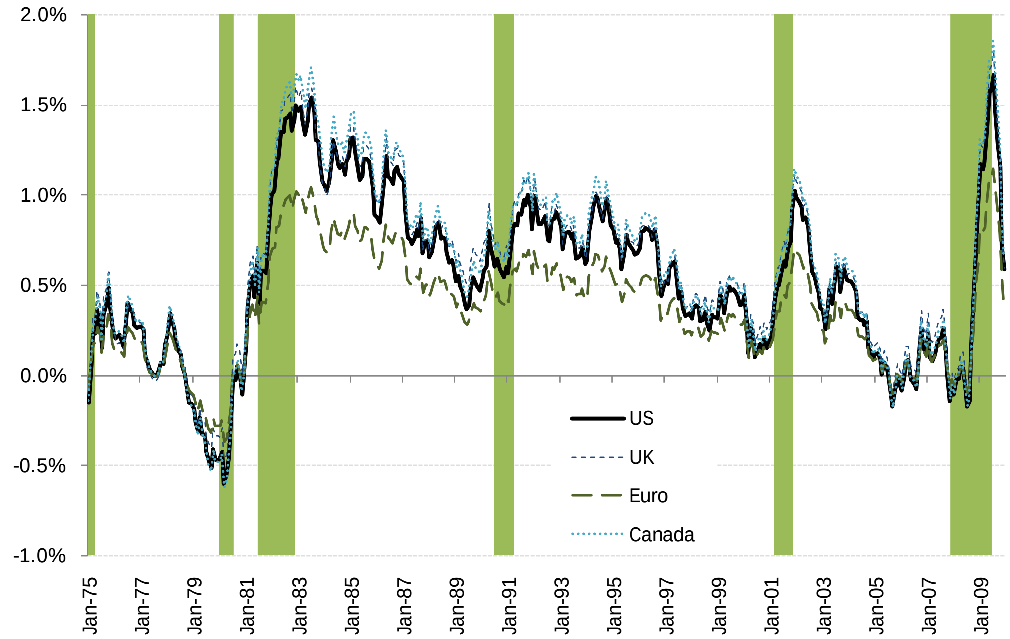
Figure 7 (cont.): Consistency with survey expectations of exchange rates



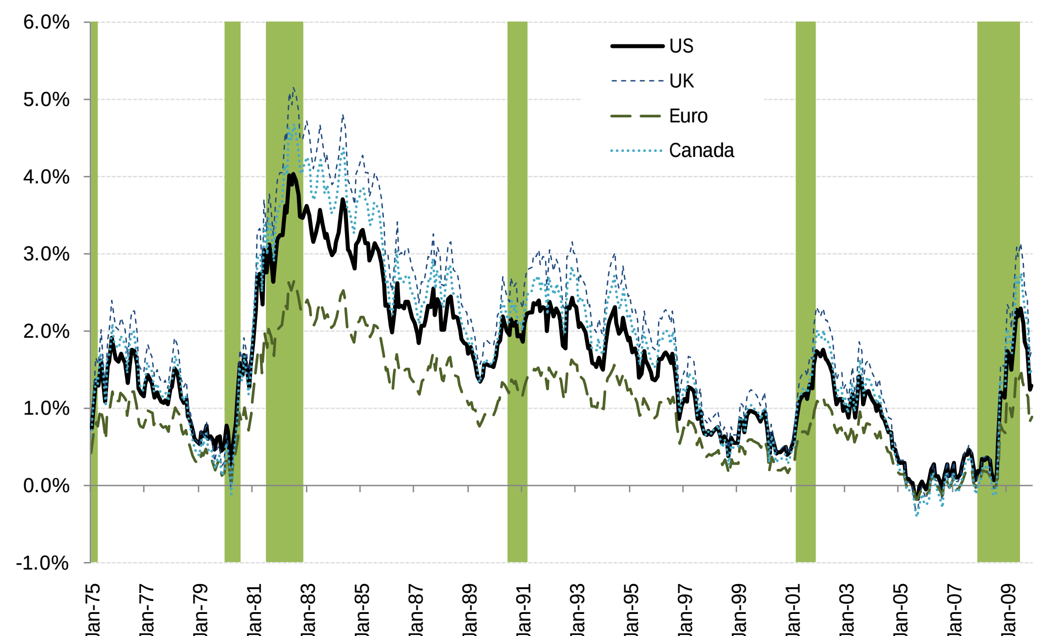
Note: The figures show the one-year ahead forecasts exchange rates generated by the unrestricted affine term structure model, the restricted affine term structure model (which includes the assumptions of global asset pricing, carry trade fundamentals and a shrinkage parameter equal to a 0.5), and the Consensus Economics survey forecasts.

Figure 8: Forward term premia on one year loans

(A) Term premia on one-year loans initiated in two years



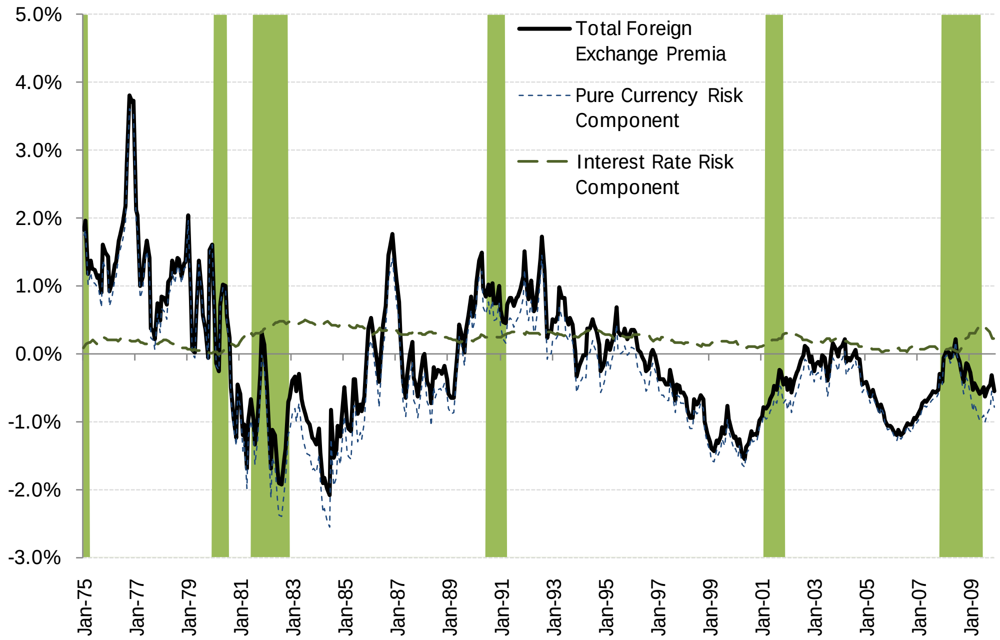
(B) Term premia on one-year loans initiated in nine years



Note: The figures show the forward term premia on "in-2-for-1" loans (one-year loans initiated in two years), $f_{j,t}^{(3)}$, and "in-9-for-1" loans (one-year loans initiated in nine years), $f_{j,t}^{(10)}$, implied by the restricted affine term structure model (which includes the assumptions of global asset pricing, carry trade fundamentals and a shrinkage parameter equal to a 0.5). Shaded areas indicate NBER recession dates for the U.S.

Figure 9: Foreign exchange risk premia

(A) USD/GBP ten-year foreign exchange premia



(B) USD/EUR ten-year foreign exchange premia

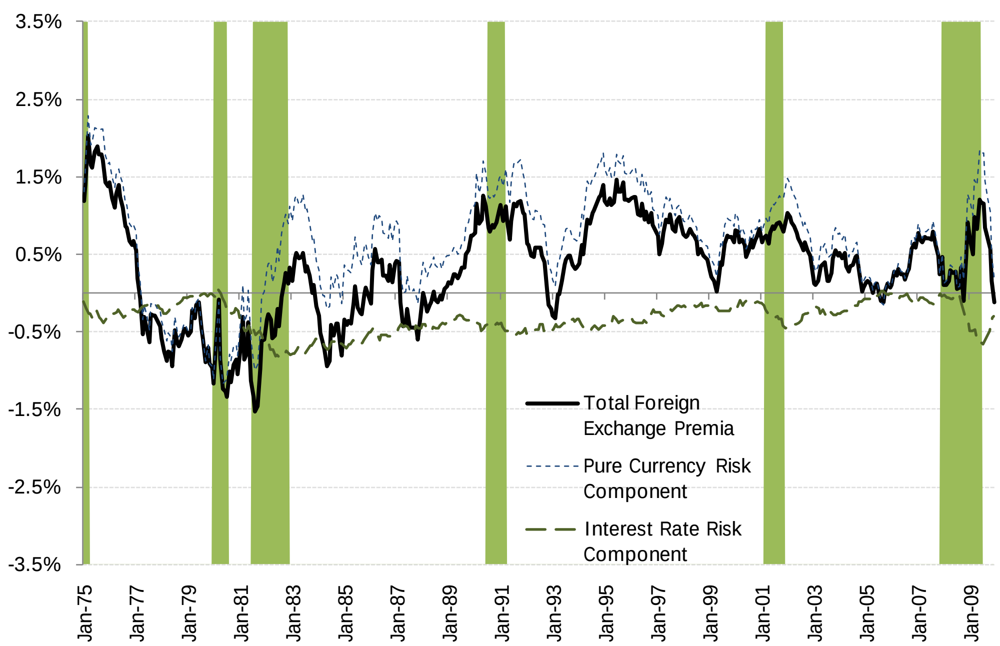
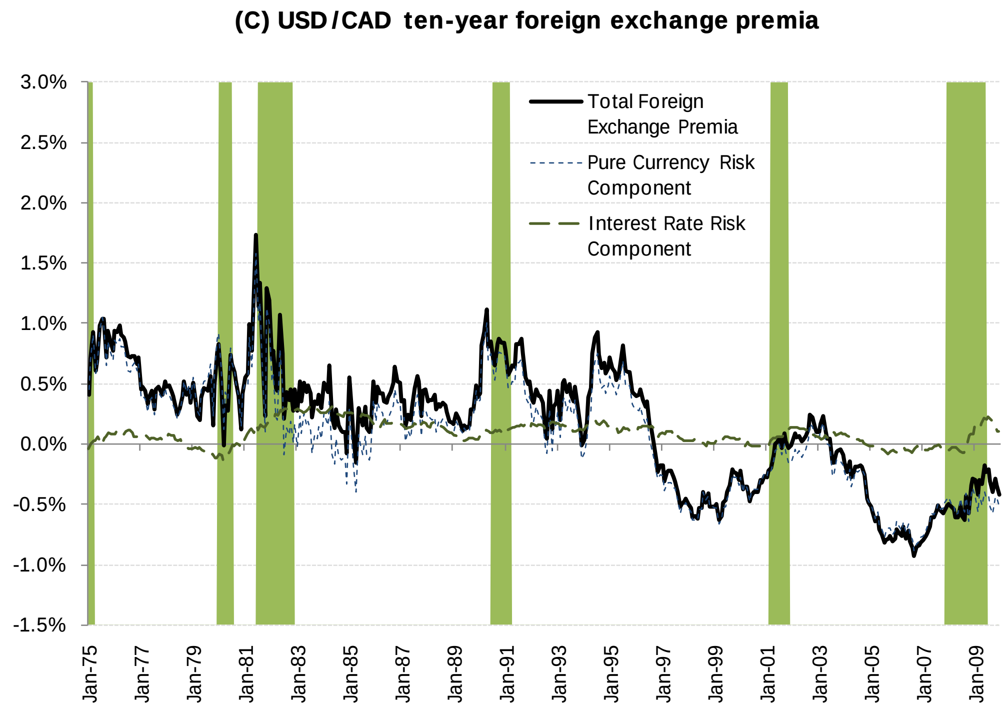


Figure 9 (cont.): Foreign exchange risk premia

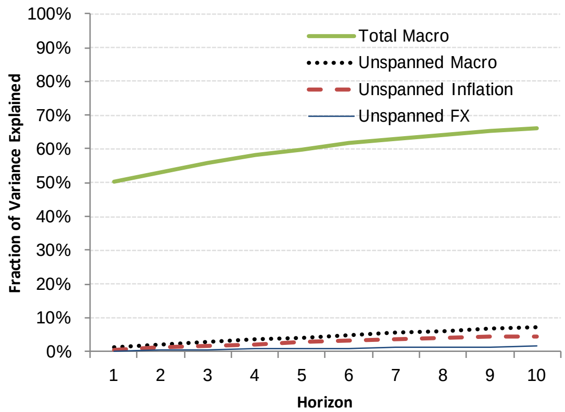


Note: The figures show the foreign exchange risk premia and their decomposition into a pure currency risk premia component and a term that reflects compensation for interest rate risk (see equation 30 in the main text) implied by the restricted affine term structure model (which includes the assumptions of global asset pricing, carry trade fundamentals and a shrinkage parameter equal to a 0.5). Shaded areas indicate NBER recession dates for the U.S.

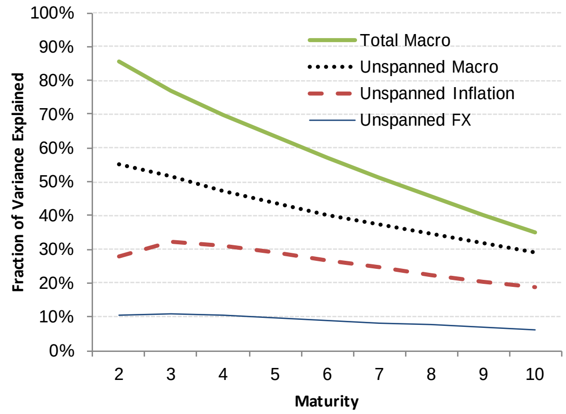
Figure 10: Effect of macroeconomic variables:
One-year ahead variance decompositions of risk premia and expectation component

(A) Bond market

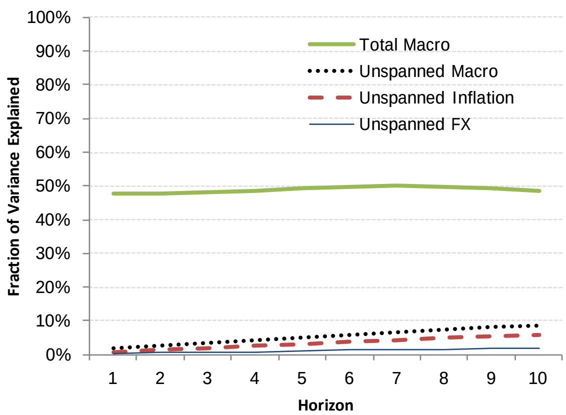
US expected short-term interest rate



US forward term premia



UK expected short-term interest rate



UK forward term premia

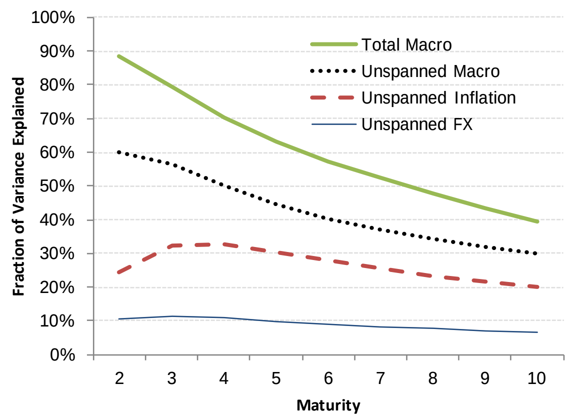
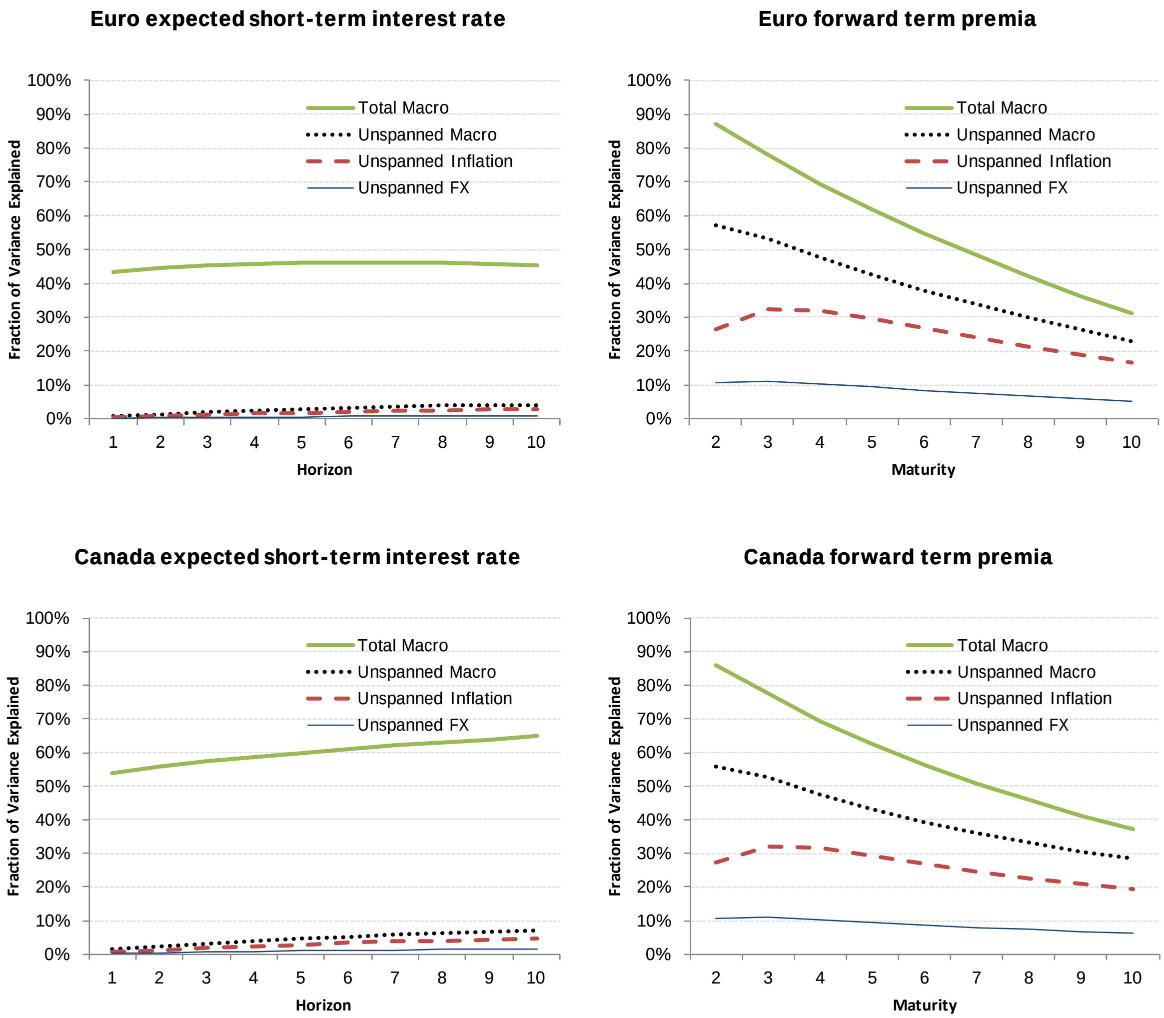


Figure 10 (cont.): Effect of macroeconomic variables:
One-year ahead variance decompositions of risk premia and expectation component

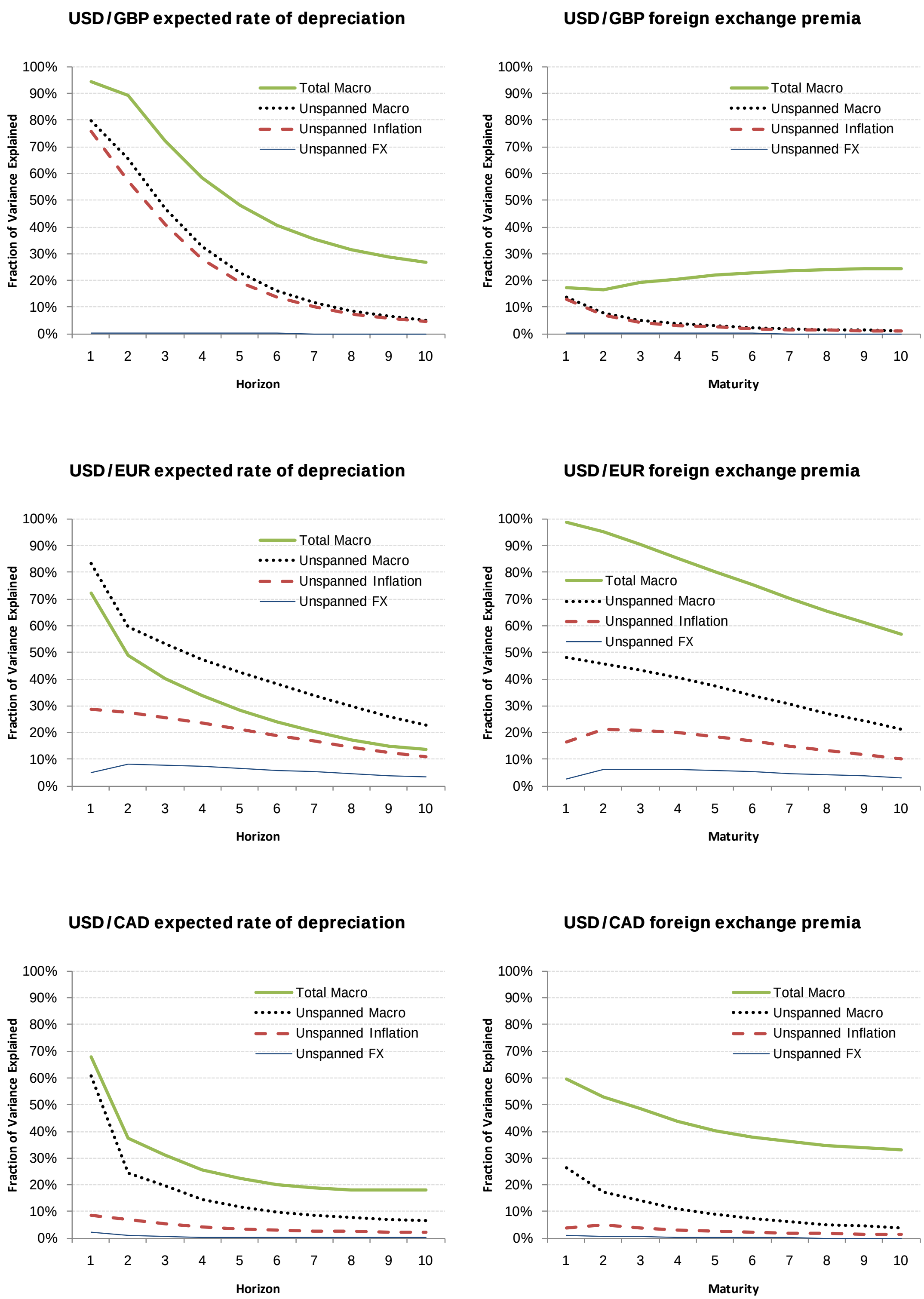
(A) Bond market (cont.)



Note: The fraction of the one-year ahead conditional variance of expected short-term interest rates and forward term premia attributable to innovations to the unspanned macro, inflation, (i.e. bonds, FX, macro ordering) and total macro components (i.e. macro first ordering) implied by the restricted affine term structure model. Unspanned inflation is the fraction of variance explained by the orthogonal component of inflation to growth and the yield curve.

Figure 10: Effect of macroeconomic variables:
One-year ahead variance decompositions of risk premia and expectation component

(B) Foreign Exchange Market



Note: The fraction of the one-year ahead conditional variance of the expected rate of depreciation and total foreign exchange risk premia attributable to innovations to the unspanned macro, inflation, (i.e. bonds, FX, macro ordering) and total macro components (i.e. macro first ordering) implied by the restricted affine term structure model. Unspanned inflation is the fraction of variance explained by the orthogonal component of inflation to growth and the yield curve.