Mixed-frequency large-scale factor models

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1 Introduction

Empirical research generally avoids the direct use of mixed frequency data by either first aggregating higher frequency series and then performing estimation and testing at the low frequency common across the series, or neglecting the low frequency data and working only on the high frequency series. The literature on large scale factor models is no exception to this practice, see e.g. Forni and Reichlin (1998), Stock and Watson (2002) and Stock and Watson (2010).

A number of mixed frequency factor models have been proposed in the literature, although they exclusively rely on small cross-sections. See for example, Mariano and Murasawa (2003), Nunes (2005), Aruoba, Diebold and Scotti (2009), Frale and Monteforte (2010), Marcellino and Schumacher (2010) and Banbura and Rünstler (2011), among others.

The purpose of this paper is to propose large scale mixed frequency factor models in the spirit of Bai and Ng (2002), Bai (2003), Bai and Ng (2006). We rely on the recent work on mixed frequency VAR models, in particular Ghysels (2012) to formulate such a model and its associated estimators. To study the large sample properties of a principal component estimation procedure, we first discuss the conditions which allow us to identify low and high frequency factors separately. The identification conditions complement those of Anderson et al. (2012) who study the identifiability of an underlying high frequency multivariate AR system from mixed frequency observations. Identifiability guarantees that the model parameters can be estimated consistently from mixed frequency data. We extend this analysis to mixed frequency factor models. Under suitable regularity conditions, the factors and loadings can be estimated via an iterative procedure which consists of estimating respectively principal components from the cross-section of high frequency data and the principal components obtained from a panel of low frequency series projected onto the high frequency factors.

An empirical application revisits the analysis of Foerster, Sarte, and Watson (2011) who use factor analytic methods to decompose industrial production (IP) into components arising from aggregate shocks and idiosyncratic sector-specific shocks. Foerster, Sarte, and Watson (2011) focus exclusively on the industrial production sectors of the US economy. Yet, IP has featured steady decline as a share of US output over the past 30 years. The US economy has become more of a service sector economy. Contrary to IP, we do not have monthly or quarterly data about the cross-section of US output across non-IP sectors, but we do on an annual basis. The US Bureau of Economic Analysis provides GDP by industry - not only IP sectors - annually. We identify two factors in a mixed frequency approximate factor model, with one being a low frequency factor pertaining to non-IP sectors. We re-examine whether the common factors reflect sectoral shocks that have propagated by way of input-output linkages between service sectors and manufacturing. Hence, our analysis completes an important part missing in the original study as it omitted a major ingredient of US economic activity. A structural factor analysis indicates that both low and high frequency aggregate shocks continue to be the dominant source of variation in the US economy. The propagation mechanism are very different, however, from those identified by Foerster, Sarte, and Watson (2011).

2 The model

2.1 Mixed frequency factor structure

Let $t = 1, 2, ..., T$ be the low frequency (LF) time units. Each period $(t-1, t]$ is divided into m subperiods with high frequency (HF) dates $t - 1 + j/m$, with $j = 1, ..., m$. For expository purpose, we present the model and the estimators in a simplified framework in which the low frequency periods are divided into two high frequency subperiods, i.e. we set $m = 2$. [2] Let $x_{1,i,t}$ and $x_{2,i,t}$, for $i = 1, ..., N_H$, be the consecutive high-frequency observations at $t - 1/2$ and t, respectively, and $y_{i,t}$, with $i = 1, ..., N_L$, the low-frequency observations at t. These observations are gathered into the N_H-dimensional vectors $x_{1,t}$, $x_{2,t}$, and the N_L-dimensional vector y_t, respectively. We assume the following linear factor structure for the stacked vector of observations:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ y_t \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & \Delta_1 \\ 0 & \Lambda & \Delta_2 \\ \Omega_1 & \Omega_2 & B \end{bmatrix} \begin{bmatrix} f_{1,t} \\ f_{2,t} \\ g_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ u_t \end{bmatrix}. \tag{1}$$

The factor structure involves two types of unobservable factors with different speeds. The first factor evolves at high frequency, and the values for subperiods 1 and 2 are denoted by $f_{1,t}$ and $f_{2,t}$, respectively. The slow factor g_t evolves at low frequency. Both types of factors can be multidimensional: the unobservable factor vectors $f_{1,t}$, $f_{2,t}$ have dimension K_H, and the unobservable factor vector g_t has di-

[2] The model with $m = 4$ high-frequency subperiods, used in the empirical application, is detailed in Appendix D.

mension K_L. In equation (1), the high frequency observations load on the high frequency factor of the same half-period via loading matrix Λ, and on the low frequency factor via loading matrices Δ_1 and Δ_2. The low frequency observations load on the high and low frequency factors via loading matrices Ω_1, Ω_2 and B, respectively. The loadings matrix Λ can feature a block structure to accommodate high frequency factors that are specific to subsets of the high frequency series. A schematic representation of the factor model is provided in Figure 1.

We assume that the loadings matrices are such that $\Lambda'\Lambda/N_H \to \Sigma_\Lambda$, as $N_H \to \infty$, $B'B/N_L \to \Sigma_B$, as $N_L \to \infty$, where Σ_Λ and Σ_B are positive definite matrices. Moreover, the idiosyncratic shocks vectors $\varepsilon_{1,t}$, $\varepsilon_{2,t}$ and u_t satisfy weak cross-sectional and serial dependence assumptions, and are assumed to be weakly correlated with the latent factors.

When $K_H = 0$, i.e. there is no high frequency factor, the specification in equation (1) reduces to a low frequency factor model with vector of observables $(x'_{1,t}, x'_{2,t}, y'_t)'$ and factor g_t, for $t = 1, 2..., T$. When $K_L = 0$, i.e. there is no low frequency factor, and $\Omega_1 = \Omega_2 = 0$, the specification in equation (1) reduces to a pure HF factor model, with observations x_τ and factor f_τ for $\tau = 1/2, 1, ..., T$, where $x_\tau = x_{1,t}$ and $f_\tau = f_{1,t}$ for $\tau = t - 1/2$, and $x_\tau = x_{2,t}$ and $f_\tau = f_{2,t}$ for $\tau = t$ and $t = 1, 2, ..., T$. Such factor specifications are considered in e.g. Stock and Watson (2002), Bai and Ng (2002) and Bai (2003) without explicit modeling of the factor dynamics, or in Forni, Hallin, Lippi, and Reichlin (2000) with explicit modeling of the factor dynamics.

As usual in latent factor models, the distribution of the factors can be normalized. First, we can assume orthogonality between $(f_{1,t}, f_{2,t})$ and g_t. Indeed, if orthogonality does not apply in a given representation of the model, factor $f_{1,t}$ can be written as the orthogonal projection on g_t plus a projection residual, i.e. $f_{1,t} = C_1 g_t + \tilde{f}_{1,t}$, where $C_1 = Cov(f_{1,t}, g_t)V(g_t)^{-1}$, and similarly $f_{2,t} = C_2 g_t + \tilde{f}_{2,t}$. Then, by plugging these equations into the model, the structure is maintained if we use $\tilde{f}_{1,t}$, $\tilde{f}_{2,t}$ and g_t as the new factors. Second, factors $f_{1,t}$, $f_{2,t}$ and g_t can be assumed to be zero-mean and standardized. Thus:

$$V\begin{pmatrix} f_{1,t} \\ f_{2,t} \\ g_t \end{pmatrix} = \begin{bmatrix} I_{K_H} & \Phi & 0 \\ \Phi' & I_{K_H} & 0 \\ 0 & 0 & I_{K_L} \end{bmatrix}, \qquad (2)$$

where Φ is the covariance between $f_{1,t}$ and $f_{2,t}$.

2.2 Factor dynamics

We complete the model specification by assuming a mixed frequency stationary Vector Autoregressive (VAR) model for the stacked vector of factors (see Ghysels (2012)). The factor dynamics is given by the following stationary structural VAR(1) model:

$$\begin{bmatrix} I_{K_H} & 0 & 0 \\ -R_H & I_{K_H} & 0 \\ 0 & 0 & I_{K_L} \end{bmatrix} \begin{bmatrix} f_{1,t} \\ f_{2,t} \\ g_t \end{bmatrix} = \begin{bmatrix} 0 & R_H & A_1 \\ 0 & 0 & A_2 \\ M_1 & M_2 & R_L \end{bmatrix} \begin{bmatrix} f_{1,t-1} \\ f_{2,t-1} \\ g_{t-1} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \\ w_t \end{bmatrix}, \tag{3}$$

where $(v'_{1,t}, v'_{2,t}, w'_t)'$ is a multivariate white noise process with mean 0 and variance-covariance matrix:

$$\Sigma = \begin{bmatrix} \Sigma_H & 0 & \Sigma_{HL,1} \\ & \Sigma_H & \Sigma_{HL,2} \\ & & \Sigma_L \end{bmatrix}. \tag{4}$$

The model accommodates coupled autoregressive dynamics for the factors at different frequencies. This coupling is induced by the sub-blocks of coefficients A_1, A_2, M_1, M_2 in the structural autoregressive matrix, and the contemporaneous correlation of factor innovations at different frequencies $\Sigma_{HL,1}$ and $\Sigma_{HL,2}$. When either $K_H = 0$ or $K_L = 0$, equation (3) implies that the latent factor follows a VAR(1) model in low or high frequency, respectively. On the other hand, if A_1, A_2, M_1, M_2 and $\Sigma_{HL,1}$, $\Sigma_{HL,2}$ are zero matrices, the high frequency and low frequency factors follow uncorrelated VAR(1) processes.

The parameters in the factor dynamics are constrained such that the sub-blocks restrictions on the unconditional variance-covariance matrix in equation (2) hold. These restrictions, derived in Appendix A, imply that each of the non-zero elements of the variance-covariance matrix Σ of the innovations, and the autocovariance matrix Φ of the high frequency factor, can be expressed in terms of parameter matrices R_H, R_L, A_1, A_2, M_1 and M_2 in the structural VAR(1) model (see Equations (A.9)-(A.14) in Appendix A). These restrictions also imply that parameters R_H, A_1 and A_2 must satisfy the following matrix equation:

$$A_1 A'_1 - R_H A_1 A'_2 - A_1 A'_2 R'_H - A_2 A'_2 = 0. \tag{5}$$

In Appendix A we also derive the stationarity conditions for the factor process.

3 Identification

In standard linear latent factor models, the normalization induced by an identity factor variance-covariance matrix identifies the factor process up to a rotation (and change of signs). Let us now show that, under suitable identification conditions, the rotation invariance of model (1) - (2) allows only for separate rotations among the components of $f_{1,t}$, among those of $f_{2,t}$, and among those of g_t. Moreover, the rotations of $f_{1,t}$ and $f_{2,t}$ are the same. Thus, the rotation invariance of model (1) - (2) maintains the interpretation of high frequency and low frequency factors, and the fact that $f_{1,t}$ and $f_{2,t}$ are consecutive observations of the same process. More formally, let us consider the following transformation of the stacked factor process:

$$\begin{bmatrix} f_{1,t} \\ f_{2,t} \\ g_t \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \tilde{f}_{1,t} \\ \tilde{f}_{2,t} \\ \tilde{g}_t \end{bmatrix} \tag{6}$$

where $(\tilde{f}'_{1,t}, \tilde{f}'_{2,t}, \tilde{g}'_t)'$ is the transformed stacked factor vector, and the block matrix $A = (A_{ij})$ is non-singular.

Definition 1. *The model is identifiable if:*

> *the data $x_{1,t},\ x_{2,t}$ and y_t satisfy a factor model of the same type as (1) and (2) with $(f'_{1,t}, f'_{2,t}, g'_t)'$ replaced by $(\tilde{f}'_{1,t}, \tilde{f}'_{2,t}, \tilde{g}'_t)'$ if, and only if, matrix A is a block-diagonal orthogonal matrix, with $A_{11} = A_{22}$.*

For the proof of identification, we distinguish two situations regarding the full-rank nature of the loading matrices.

3.1 Identification under full-rank conditions

Proposition 1. *Assume that matrix Λ is full column rank and that*

$$\text{either matrix } [\Lambda \vdots \Delta_1], \text{ or matrix } [\Lambda \vdots \Delta_2], \text{ is full column rank (for } N_H \text{ large enough).} \tag{7}$$

Then, the model is identifiable.

The proof of Proposition 1 is given in Appendix B. The full-rank condition for the loadings matrix is a standard assumption in linear factor models (see e.g. Assumption B in Bai and Ng (2002) and Bai (2003)). In Proposition 1, it is enough that the full-rank condition applies to at least one of the high frequency panels.

3.2 Identification with reduced-rank loading matrices

When the loading matrices $[\Lambda \vdots \Delta_1]$ and $[\Lambda \vdots \Delta_2]$ in the DGP are both reduced-rank, we cannot apply Proposition 1 to show identification. This situation applies for instance when the high frequency data do not load on the low frequency factors. We maintain the hypothesis that matrix Λ is full-rank (for N_H large enough), and focus on the case of a single low frequency factor, i.e. $K_L = 1$. Then, a reduced-rank problem occurs if both vectors Δ_1 and Δ_2 are spanned by the columns of matrix Λ, that is

$$\Delta_1 = \Lambda d_1, \text{ and } \Delta_2 = \Lambda d_2, \tag{8}$$

for some vectors d_1 and d_2. Then, using the tranformation in Equation (6) the model can be written as:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ y_t \end{bmatrix} = \begin{bmatrix} \tilde{\Lambda} & 0 & 0 \\ 0 & \tilde{\Lambda} & 0 \\ \tilde{\Omega}_1 & \tilde{\Omega}_2 & \tilde{B} \end{bmatrix} \begin{bmatrix} \tilde{f}_{1,t} \\ \tilde{f}_{2,t} \\ \tilde{g}_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ u_t \end{bmatrix}, \tag{9}$$

where the transformed factors

$$\begin{cases} \tilde{f}_{1,t} &= (I_{K_H} + d_1 d_1')^{-1/2}(f_{1,t} + d_1 g_t) \\ \tilde{f}_{2,t} &= (I_{K_H} + d_2 d_2')^{-1/2}(f_{2,t} + d_2 g_t) \\ \tilde{g} &= (1 + d_1' d_1 + d_2' d_2 + 2 d_1' \Phi d_2)^{-1/2}(g_t - d_1' f_{1,t} - d_2' f_{2,t}) \end{cases}, \tag{10}$$

satisfy the normalization restriction (2) with a transformed autocovariance matrix $\tilde{\Phi}$, and $\tilde{\Lambda}$, $\tilde{B}$, $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are transformed matrices of loadings. Thus, the model can be rewritten as a model without the effect of the low frequency factor on the high frequency observations, i.e. $\tilde{\Delta}_1 = \tilde{\Delta}_2 = 0$, by suitably redefining the high and low frequency factors. To eliminate this multiplicity of representations, we introduce the following restriction:

Assumption 1. *Let $K_L = 1$. If vectors Δ_1 and Δ_2 are spanned by Λ, then $\Delta_1 = \Delta_2 = 0$.*

The next Proposition shows that this identification condition is sufficient to identify the model.

Proposition 2. *Let $K_L = 1$ and $\Delta_1 = \Delta_2 = 0$ in the DGP. Then, the model is identifiable.*

3.3 Normalization of factor loadings

When the model is identifiable in the sense of Definition 1, we can eliminate the rotation invariance of high frequency and low frequency factors as in standard latent factor models (see, e.g., Bai and Ng (2013) for a thorough discussion of identification in latent factor models). In this paper we impose the diagonality of the variance-covariance matrices of the loadings:

$$\Sigma_\Lambda = diag(\sigma^2_{\lambda,k}), \qquad \Sigma_B = diag(\sigma^2_{b,k}).$$

Then, the high frequency and low frequency factor processes are identifiable up to a change of signs.

4 Estimation

4.1 The estimators of the factor values

The estimates of the factor values are obtained by an iterative estimation procedure. At each iteration the HF and LF factors are estimated in two separate steps by Principal Component Analysis (PCA) applied to suitable matrices of HF and LF residuals. The main idea is that from the model in equation (1) residuals $x_{j,t} - \Delta_j g_t$ satisfy a factor model with factor $f_{j,t}$ in high frequency, and residuals $y_t - \Omega_1 f_{1,t} - \Omega_2 f_{2,t}$ satisfy a factor model with factor g_t in low frequency.

The iteration p consists in the following two steps:

1. Define $\hat{G}^{(p-1)} = [\tilde{g}_1, \ ..., \ \tilde{g}_T]'$ as the $(T \times K_L)$ matrix of estimated LF factors obtained in the previous iteration. Regress each sub-panel of the HF observations on $\hat{G}^{(p-1)}$ to obtain the estimated loadings matrices $\hat{\Delta}_1$ and $\hat{\Delta}_2$, and the residuals:

$$\hat{\xi}_{j,t} = x_{j,t} - \hat{\Delta}_j \tilde{g}_t \ , \qquad j = 1, 2 \ .$$

Collecting the residuals in the $(2T \times N_H)$ matrix:

$$\hat{\Xi} = [\hat{\xi}_{1,1}, \ \hat{\xi}_{2,1}, \ ..., \ \hat{\xi}_{1,T}, \hat{\xi}_{2,T}]',$$

the $(2T \times K_H)$ matrix $\hat{F}^{(p)} = [\hat{f}_{1,1}, \hat{f}_{2,1}, ..., \hat{f}_{1,T}, \hat{f}_{2,T}]'$ of estimated HF factor values is obtained by PCA:

$$\left(\frac{1}{2N_H T} \hat{\Xi}\hat{\Xi}' \right) \hat{F}^{(p)} = \hat{F}^{(p)} \hat{V}_F, \tag{11}$$

where $\hat{V}_F$ is the diagonal matrix of the eigenvalues. The estimated HF loadings matrix $\hat{\Lambda}$ is obtained from the high frequency least squares regression of x_τ on factor $\hat{f}_\tau$ for $\tau = 1/2, 1, ..., T$, where $x_\tau = x_{1,t}$ and $\hat{f}_\tau = \hat{f}_{1,t}$ for $\tau = t - 1/2$, and $x_\tau = x_{2,t}$ and $f_\tau = \hat{f}_{2,t}$ for $\tau = t$ and $t = 1, 2, ..., T$.

2. Define:

$$\hat{F}^{*(p)} = \left[\ \hat{F}_1^{(p)} \ \vdots \ \hat{F}_2^{(p)} \ \right] = \begin{bmatrix} \hat{f}'_{1,1} & \hat{f}'_{2,1} \\ \vdots & \vdots \\ \hat{f}'_{1,T} & \hat{f}'_{2,T} \end{bmatrix},$$

as the $(T \times 2K_H)$ matrix of estimated HF factors obtained in the previous step, where the factor values of the two subperiods are stacked horizontally. Regress the LF observations y on $\hat{F}^{*(p)}$ to obtain the $(T \times N_L)$ matrix of residuals:

$$\hat{\Psi} = [\hat{\psi}_1, \ ..., \ \hat{\psi}_T]'$$

where:

$$\hat{\psi}_t = y_t - \hat{\Omega}_1 \hat{f}_{1,t} - \hat{\Omega}_2 \hat{f}_{2,t}, \qquad t = 1, ..., T$$

with $\hat{\Omega}_1$ and $\hat{\Omega}_2$ being the matrices of estimated loadings. The estimated LF factors $\hat{G}^{(p)} = [\hat{g}_1, ..., \hat{g}_T]'$ are obtained performing PCA:

$$\left(\frac{1}{N_L T} \hat{\Psi} \hat{\Psi}' \right) \hat{G}^{(p)} = \hat{G}^{(p)} \hat{V}_G, \tag{12}$$

where $\hat{V}_G$ is the diagonal matrix of the eigenvalues. The estimated LF loadings matrix $\hat{B}$ is obtained from the low frequency least squares regression of y_t on $\hat{g}_t$. By construction, the estimated factors $\hat{g}_t$ are orthogonal to $(\hat{f}'_{1,t}, \hat{f}'_{2,t})'$.

The procedure is iterated replacing $\hat{G}^{(p-1)}$ with $\hat{G}^{(p)}$ in step 1 and can be initialized performing the PCA in step 1 with $\hat{\xi}_{j,t} = x_{j,t}$, i.e. with $\hat{G}^{(0)} = 0$.

4.2 Estimation of the factor dynamics

The free parameters of the factor dynamics can be estimated by using the reduced form of the VAR(1) model in equation (3) and replacing the unobservable factor values with their estimates obtained in Section 4.1. The reduced form of the VAR(1) model in equation (3) is given by (see Ghysels (2012)):

$$\begin{bmatrix} f_{1,t} \\ f_{2,t} \\ g_t \end{bmatrix} = \begin{bmatrix} 0 & R_H & A_1 \\ 0 & R_H^2 & R_H A_1 + A_2 \\ M_1 & M_2 & R_L \end{bmatrix} \begin{bmatrix} f_{1,t-1} \\ f_{2,t-1} \\ g_{t-1} \end{bmatrix} + \zeta_t, \tag{13}$$

where ζ_t is a zero-mean white noise process with variance-covariance matrix Σ_ζ given in Equation (A.17) in Appendix A.2. Let us denote by $\theta \in \mathbb{R}^p$, say, the parameter vector collecting the elements in the matrices A_1, A_2, R_H, R_L, M_1 and M_2. By using the normalization restrictions on the factor process given in Equations (A.9)-(A.14), matrix Σ_ζ in Equation (A.17) can be written in terms of vector θ, i.e. $\Sigma_\zeta = \Sigma_\zeta(\theta)$. Then, the reduced-form factor dynamics in Equation (13) becomes:

$$z_t = C(\theta) z_{t-1} + \zeta_t, \tag{14}$$

where $z_t = [f'_{1,t}, f'_{2,t}, g'_t]'$ is the vector of stacked factors, matrix $C(\theta)$ is the autoregressive matrix in Equation (13) written as a function of θ, and $V(\zeta_t) = \Sigma_\zeta(\theta)$. The parameter θ is subject to the constraint $\theta \in \Theta$, where $\Theta \subset \mathbb{R}^p$ is the set of parameters values that satisfy matrix equation (5). We estimate parameter θ by constrained Gaussian Pseudo Maximum Likelihood (PML) by replacing the unobserved factor values $f_{1,t}$, $f_{2,t}$ and g_t with their estimates $\hat{f}_{1,t}$, $\hat{f}_{2,t}$ and $\hat{g}_t$ for all $t = 1, ..., T$. The estimator of parameter θ is:

$$\hat{\theta} = \arg\max_{\theta \in \Theta} \hat{Q}_T(\theta), \tag{15}$$

where the criterion $\hat{Q}_T(\theta)$ is the Gaussian log-likelihood function:

$$\hat{Q}_T(\theta) = -\frac{1}{2} \log |\Sigma_\zeta(\theta)| - \frac{1}{2T} \sum_{t=2}^{T} [\hat{z}_t - C(\theta)\hat{z}_{t-1}]' \, \Sigma_\zeta(\theta)^{-1} [\hat{z}_t - C(\theta)\hat{z}_{t-1}], \tag{16}$$

and involves the factor estimates.

5 Large sample properties of the estimators

Let us assume that the HF and LF factors are one-dimensional, i.e. $K_H = K_L = 1$. The next Proposition provides the linearization of the iterative estimators defined by equations (11) and (12) around the true factor values.

Proposition 3. *For large N_H, N_L and T, the estimators $\hat{F}^{(p)}$ and $\hat{G}^{(p)}$ satisfy the linearized iteration step:*

$$\begin{aligned} \hat{F}^{(p)} \hat{h}_F^{-1} - F &= \eta_F + \mathscr{L}_F (\hat{G}^{(p-1)} \hat{h}_G^{-1} - G), \\ \hat{G}^{(p)} \hat{h}_G^{-1} - G &= \eta_G + \mathscr{L}_G (\hat{G}^{(p-1)} \hat{h}_G^{-1} - G), \end{aligned}$$

for some random positive scalars $\hat{h}_F$ and $\hat{h}_G$, where the random vectors η_F and η_G are such that $\|\eta_F\|/\sqrt{T} = O_p(T^{-1/2})$ and $\|\eta_G\|/\sqrt{T} = O_p(T^{-1/2})$, and $F = (F'_1, F'_2)'$ and G are the $(2T \times 1)$ and $(T \times 1)$ vectors of the true values of the HF and LF factors. The $(T \times T)$ matrix $\mathscr{L}_G$ has (asymptotically) the eigenvalues:

- *0, associated with the eigenvector G,*

- *1, with multiplicity 2, associated with the eigenspace spanned by* $F_1 + 2(w_1 + \phi w_2)G$ *and* $F_2 + 2(w_1\phi + w_2)G$*,*

- $w_1 d_1 + w_2 d_2$*, with multiplicity* $T - 3$*, associated with the eigenspace that is the orthogonal complement of the linear space spanned by* F_1*,* F_2 *and* G*.*

The constants w_1*,* w_2*,* d_1 *and* d_2 *are defined as:*

$$w_j = \lim_{N_L \to \infty} \left(\frac{B'B}{N_L} \right)^{-1} \left(\frac{B'\Omega_j}{N_L} \right), \qquad d_j = \lim_{N_H \to \infty} \left(\frac{\Lambda'\Lambda}{N_H} \right)^{-1} \left(\frac{\Lambda'\Delta_j}{N_H} \right), \qquad j = 1, 2,$$

and $\phi = Cov(f_{1,t}, f_{2,t})$ *is the stationary autocorrelation of the HF factor.*

The proof of Proposition 3 is given in Appendix C.

Proposition 4 provides the consistency of the factor values estimates at rate $\sqrt{T}$. We use the root mean squared error criterion to assess convergence of the factor estimates at different dates.

Proposition 4. *Assuming* $N_H, N_L, T \to \infty$*, s.t.* $N_H \gg N_L \geq T$ *and other regularity conditions:*

$$T^{-1/2} \| \hat{F} \hat{h}_F^{-1} - F \| + T^{-1/2} \| \hat{G} \hat{h}_G^{-1} - G \| = O_p \left(\frac{1}{\sqrt{T}} \right),$$

where F *and* G *are the vectors of the true factor values.*

The proof of Proposition 4 is given in Appendix C.

Proposition 5. *Assuming* $N_H, N_L, T \to \infty$*, s.t.* $N_H \gg N_L \geq T$ *and other regularity conditions:*

$$\| \hat{\theta} - \theta \| = O_p \left(\frac{1}{\sqrt{T}} \right).$$

The proof of Proposition 5 is given in Appendix C.

6 Monte Carlo analysis

[...]

7 Empirical application

[...]

8 Conclusions

[...]

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TABLES

Table 1: Estimated number of factors (HF data: IP indexes, LF: non-IP **real value added GDP**)

IC_{p1}: growth rates of indexes

$[Y\,X_1]$	$[Y\,X_2]$	$[Y\,X_3]$	$[Y\,X_4]$	$[Y\,X_{1:4}]$	$[Y\,X_{LF}]$	$[X_{LF}]$	$[X_{HF}]$	$[Y]$
1	2	1	2	3	2	2	2	1

IC_{p2}: growth rates of indexes

$[Y\,X_1]$	$[Y\,X_2]$	$[Y\,X_3]$	$[Y\,X_4]$	$[Y\,X_{1:4}]$	$[Y\,X_{LF}]$	$[X_{LF}]$	$[X_{HF}]$	$[Y]$
1	2	1	1	3	1	2	1	1

IC_{p1}: **innovations to sectoral productivity** (ε_t in Foerster, Sarte, and Watson (2011))

$[\varepsilon_Y\,\varepsilon_{X1}]$	$[\varepsilon_Y\,\varepsilon_{X2}]$	$[\varepsilon_Y\,\varepsilon_{X3}]$	$[\varepsilon_Y\,\varepsilon_{X4}]$	$[\varepsilon_Y\,\varepsilon_{X1:4}]$	$[\varepsilon_Y\,\varepsilon_{X,LF}]$	$[\varepsilon_{X,LF}]$	$[\varepsilon_{X,HF}]$	$[\varepsilon_Y]$
1	2	1	1	1	2	3	2	1

IC_{p2}: **innovations to sectoral productivity** (ε_t in Foerster, Sarte, and Watson (2011))

$[\varepsilon_Y\,\varepsilon_{X,1}]$	$[\varepsilon_Y\,\varepsilon_{X,2}]$	$[\varepsilon_Y\,\varepsilon_{X,3}]$	$[\varepsilon_Y\,\varepsilon_{X,4}]$	$[\varepsilon_Y\,\varepsilon_{X,1:4}]$	$[\varepsilon_Y\,\varepsilon_{X,LF}]$	$[\varepsilon_{X,LF}]$	$[\varepsilon_{X,HF}]$	$[\varepsilon_Y]$
1	1	1	1	1	1	2	1	1

In the table we display the estimated number of latent factors for different panels of mixed frequency data, using the information criteria IC_{p1} and IC_{p2} proposed by Bai and Ng (2002). In the first 2 lines, the two panels of observable variables have the following dimensions: $N_H = 117$, $N_L = 42$, $T = 35$. The notation $[YX_i]$ indicates that panel Y and panel X_i are stacked together in a unique panel, and the number of latent factors is determined in this new panel. Y denotes the panel of LF (yearly) observations of growth rates of real **value added GDP** for the sample period 1977-2011, for the following 42 sectors: 35 services, Construction, Farms, Forestry-Fishing and related activities, General government (federal), Government enterprises (federal), General government (states and local) and Government enterprises (states and local). X_i denotes the panel of HF (quarterly) observations of growth rates for the sample period 1977.Q1-2011.Q4, for the 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), for quarter i, for $i = 1, 2, 3, 4$. X_{HF} denotes the $4T \times N_H$ panel of HF observations for all quarters in the sample. X_{LF} denotes the panel of HF observations of growth rates aggregated as a panel of LF observations (the aggregation is performed by taking the mean of the quarterly observations). $X_{1:4}$ denotes the $T \times 4N_H$ panel of HF observations for all quarters in the sample, with observations of different quarters stacked along the columns. In our model, the number of factors is $K_L + K_H$ for panels $[YX_i]$, $i = 1, 2, 3, 4$, $K_L + 4K_H$ for panel $[YX_{1:4}]$ and $K_L + K_H$ for panel $[YX_{LF}]$.

In the third and fourth line we perform the same type of analysis as in the first two lines, but on the panels of sectoral productivity shocks (ε_t in Foerster, Sarte, and Watson (2011)). ε_X denotes the panel of productivity shocks for the 117 IP sectors, and ε_Y denotes the panel of productivity shocks for the panel of 38 non-manufacturing sectors (corresponding to the 42 considered before, excluding the 4 Government related sectors, as capital flows data are not available for these sectors.).

TABLE 1 BIS: Estimated number of factors (HF data: IP indexes, LF: non-IP real **GROSS OUTPUT**)

IC_{p1}: growth rates of indexes

$[Y\ X_1]$	$[Y\ X_2]$	$[Y\ X_3]$	$[Y\ X_4]$	$[Y\ X_{1:4}]$	$[Y\ X_{LF}]$	$[X_{LF}]$	$[X_{HF}]$	$[Y]$
1	1	2	2	2	2	2	1	15

IC_{p2}: growth rates of indexes

$[Y\ X_1]$	$[Y\ X_2]$	$[Y\ X_3]$	$[Y\ X_4]$	$[Y\ X_{1:4}]$	$[Y\ X_{LF}]$	$[X_{LF}]$	$[X_{HF}]$	$[Y]$
1	1	2	2	2	2	1	1	2

IC_{p1}: **innovations to sectoral productivity** (ε_t in Foerster, Sarte, and Watson (2011))

$[\varepsilon_Y\ \varepsilon_{X1}]$	$[\varepsilon_Y\ \varepsilon_{X2}]$	$[\varepsilon_Y\ \varepsilon_{X3}]$	$[\varepsilon_Y\ \varepsilon_{X4}]$	$[\varepsilon_Y\ \varepsilon_{X1:4}]$	$[\varepsilon_Y\ \varepsilon_{X,LF}]$	$[\varepsilon_{X,LF}]$	$[\varepsilon_{X,HF}]$	$[\varepsilon_Y]$
1	1	1	1	1	2	3	1	15

IC_{p2}: **innovations to sectoral productivity** (ε_t in Foerster, Sarte, and Watson (2011))

$[\varepsilon_Y\ \varepsilon_{X,1}]$	$[\varepsilon_Y\ \varepsilon_{X,2}]$	$[\varepsilon_Y\ \varepsilon_{X,3}]$	$[\varepsilon_Y\ \varepsilon_{X,4}]$	$[\varepsilon_Y\ \varepsilon_{X,1:4}]$	$[\varepsilon_Y\ \varepsilon_{X,LF}]$	$[\varepsilon_{X,LF}]$	$[\varepsilon_{X,HF}]$	$[\varepsilon_Y]$
1	1	1	1	1	1	1	1	1

In the table we display the estimated number of latent factors for different panels of mixed frequency data, using the information criteria IC_{p1} and IC_{p2} proposed by Bai and Ng (2002). In the first 2 lines, the two panels of observable variables have the following dimensions: $N_H = 117$, $N_L = 38$, $T = 24$. The notation $[Y X_i]$ indicates that panel Y and panel X_i are stacked together in a unique panel, and the number of latent factors is determined in this new panel. Y denotes the panel of LF (yearly) observations of growth rates of real **GROSS OUTPUT** for the sample period 1988-2011, for the following 38 sectors: 35 services, Construction, Farms, Forestry-Fishing and related activities. X_i denotes the panel of HF (quarterly) observations of growth rates for the sample period 1988.Q1-2011.Q4, for the 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), for quarter i, for $i = 1, 2, 3, 4$. X_{HF} denotes the $4T \times N_H$ panel of HF observations for all quarters in the sample. X_{LF} denotes the panel of HF observations of growth rates aggregated as a panel of LF observations (the aggregation is performed by taking the mean of the quarterly observations). $X_{1:4}$ denotes the $T \times 4N_H$ panel of HF observations for all quarters in the sample, with observations of different quarters stacked along the columns. In our model, the number of factors is $K_L + K_H$ for panels $[Y X_i]$, $i = 1, 2, 3, 4$, $K_L + 4K_H$ for panel $[Y X_{1:4}]$ and $K_L + K_H$ for panel $[Y X_{LF}]$.

In the third and fourth line we perform the same type of analysis as in the first two lines, but on the panels of sectoral productivity shocks (ε_t in Foerster, Sarte, and Watson (2011)). ε_X denotes the panel of productivity shocks for the 117 IP sectors, and ε_Y denotes the panel of productivity shocks for the panel of 38 non-manufacturing sectors. For productivity innovations, T = 23, as the innovation for the first year in the sample cannot be computed.

Table 2: Regressions of HF and LF observables on **1 HF and 1 LF factors: quantiles of adjusted** R^2 **(HF data: IP indexes, LF: non-IP real value added GDP).**

a) $\bar{R}^2$ Quantile. OBSERVABLES: growth rates of indexes. FACTORS: extracted from original data.

Obs.	Factors	10%	25%	50%	75%	90%
Y	LF	-3.0	-2.3	4.4	11.2	22.0
Y	LF, HF	0.5	8.7	24.5	46.0	56.1
Y	HF	-4.0	3.9	15.8	33.2	45.2
X	LF	-2.5	-2.0	-1.2	0.4	2.2
X	HF, LF	0.7	5.9	24.8	38.2	57.1
X	HF	-0.0	5.3	25.8	38.3	57.0

b) $\bar{R}^2$ Quantile. OBSERVABLES: ε_X and ε_Y, i.e. sectoral productivity innovations (ε_t in Foerster, Sarte, and Watson (2011)). FACTORS: extracted from sectoral productivity innovations.

Obs.	Factors	10%	25%	50%	75%	90%
ε_Y	LF	-2.8	-1.6	5.9	11.9	26.7
ε_Y	LF, HF	-0.5	7.4	12.5	38.3	49.0
ε_Y	HF	-3.9	1.0	4.7	18.9	35.8
ε_X	LF	-2.0	-1.4	-0.6	1.6	3.3
ε_X	HF, LF	-0.9	2.3	9.9	22.5	40.8
ε_X	HF	-0.6	1.0	8.2	20.1	40.6

c) $\bar{R}^2$ Quantile. OBSERVABLES: growth rates of indexes. **FACTORS: extracted from sectoral productivity innovations**.

Obs.	Factors	10%	25%	50%	75%	90%
Y	LF	-2.7	-0.4	5.3	17.0	29.8
Y	LF, HF	3.4	6.4	16.2	48.5	60.2
Y	HF	-2.3	2.7	9.5	23.6	38.2
X	LF	-2.1	-1.2	0.6	2.2	5.0
X	HF, LF	0.9	4.2	21.3	35.8	52.4
X	HF	-0.1	3.0	20.2	32.2	49.2

d) $\bar{R}^2$ Quantile. **OBSERVABLES: growth rates of indexes. FACTORS: extracted from sectoral productivity innovations and their lagged values (only lag 1 is considered)**.

Obs.	Factors	10%	25%	50%	75%	90%
Y	LF	-2.6	0.7	10.0	22.4	36.3
Y	LF, HF	2.7	10.2	23.3	54.1	63.1
Y	HF	-4.0	0.1	12.1	28.7	49.7
X	LF	-3.7	-2.7	-0.4	2.4	5.3
X	HF, LF	-0.3	5.7	23.8	40.8	60.0
X	HF	0.2	4.4	21.7	37.7	56.7

TABLE 2: description of dataset and methodology (HF data: IP indexes, LF: non-IP real **value added GDP**)

In the table we display the quantiles of the empirical distributions of the adjusted R^2, denoted $\bar{R}^2$, for different sets of time series regressions.

Panel a)
The regressions in the first three lines involve the real GDP growth rates of the 42 sectors (35 services, Construction, Farms, Forestry-Fishing and related activities + 4 Government sectors) as dependent variables, while the regressions in the last tree lines involve the growth of the 117 industrial production indexes as dependent variables. The factors used as explanatory variables are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model (**MFFM**) with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the regressions is 1977.Q1-2011.Q4. In lines 1 and 4 we report the quantiles of $\bar{R}^2$ of the regressions using as explanatory variable the estimated LF factor only. In lines 2 and 5 we report the quantiles of $\bar{R}^2$ of the regressions using as explanatory variables the estimated LF and HF factors. In lines 3 and 6 we report the quantiles of $\bar{R}^2$ of the regressions using as explanatory variable the estimated HF factor only. The regressions in lines 2 and 3 are unrestricted MIDAS regressions. The regressions in lines 4 and 5 allow the estimated coefficients of the LF factor to be different at each quarter.

Panel b)
The regressions in the first three lines involve the **productivity innovations of the 38 non-IP sectors (35 services, Construction, Farms, Forestry-Fishing and related activities) as dependent variables**, while the regressions in the last tree lines involve the **productivity innovations of the 117 industrial production indexes as dependent variables**. The factors used as explanatory variables are estimated from the panel of productivity innovations computed as proposed by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model (**MFFM**) with $K_H = K_L = 1$. **The sample period for the estimation of both the factor model and the regressions is 1978.Q1-2011.Q4, because the productivity shocks can not be computed for the first year of the sample (see Foerster, Sarte, and Watson (2011), especially their equation (B38) on page 10 of their Appendix B.)**

Panel c)
The regressions in the first three lines involve the real GDP growth of the 38 non-IP sectors (35 services, Construction, Farms, Forestry-Fishing and related activities) as dependent variables, while the regressions in the last tree lines involve the growth of the 117 industrial production indexes as dependent variables. The factors used as explanatory variables are estimated from the panel of productivity innovations computed as proposed by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model (**MFFM**) with $K_H = K_L = 1$. **The sample period for the estimation of both the factor model and the regressions is 1978.Q1-2011.Q4.**

Panel d)
The regressions in the first three lines involve the real GDP growth of 38 non-IP sectors (35 services, Construction, Farms, Forestry-Fishing and related activities) as dependent variables, while the regressions in the last tree lines involve the growth of the 117 industrial production indexes as dependent variables. The factors used as explanatory variables are estimated from the panel of productivity innovations computed as proposed by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model (**MFFM**) with $K_H = K_L = 1$. **Both the contemporaneous and lagged values (only lag 1 is included) of the factors are used as explanatory variables. The choice of including the lags of the factors as regressors, is justified by equations (10) and (12) in Foerster, Sarte, and Watson (2011)**. The sample period for the estimation of both the factor model and the regressions is 1979.Q1-2011.Q4.

TABLE 2 **BIS**: Regressions of HF and LF observables on 1 HF and 1 LF factors: quantiles of adjusted R^2. (HF data: IP indexes, LF: non-IP real **GROSS OUTPUT**).

a) $\bar{R}^2$ Quantile. OBSERVABLES: indexes growth rates (Y are **Gross Output** growth rates). FACTORS: extracted from original data.

Obs.	Factors	10%	25%	50%	75%	90%
Y	LF	-3.6	-0.3	4.8	23.6	31.0
Y	LF, HF	-2.4	28.9	45.3	66.0	80.2
Y	HF	-3.0	7.0	28.0	44.0	63.0
X	LF	-3.6	-3.2	-2.2	-0.6	1.9
X	HF, LF	-1.4	5.7	22.6	40.1	63.2
X	HF	0.4	5.1	21.8	41.2	63.2

b) $\bar{R}^2$ Quantile. OBSERVABLES: ε_X and ε_Y, i.e. sectoral productivity innovations (ε_t in Foerster, Sarte, and Watson (2011)). FACTORS: extracted from sectoral productivity innovations.

Obs.	Factors	10%	25%	50%	75%	90%
Y	LF	-4.4	-3.7	-1.3	13.0	34.3
Y	LF, HF	-13.1	14.4	32.9	53.4	62.7
Y	HF	-9.2	-0.8	20.7	38.1	52.8
X	LF	-3.9	-3.1	-1.5	0.7	4.0
X	HF, LF	-2.9	0.3	7.8	19.3	34.6
X	HF	-1.0	0.6	4.8	19.6	35.9

c) $\bar{R}^2$ Quantile. OBSERVABLES: indexes growth rates (Y are **Gross Output** growth rates). FACTORS: extracted from sectoral productivity innovations.

Obs.	Factors	10%	25%	50%	75%	90%
Y	LF	-4.2	-0.2	4.7	21.0	43.8
Y	LF, HF	2.2	20.2	44.7	66.4	81.1
Y	HF	-3.1	5.8	20.7	42.9	65.4
X	LF	-4.0	-3.1	-1.7	-0.2	3.5
X	HF, LF	-2.1	2.4	19.7	38.0	54.6
X	HF	-0.3	2.8	19.4	35.8	53.0

d) $\bar{R}^2$ Quantile. OBSERVABLES: indexes growth rates (Y are **Gross Output** growth rates). FACTORS: extracted from sectoral productivity innovations and their lagged values (only lag 1 is considered).

Obs.	Factors	10%	25%	50%	75%	90%
Y	LF	-8.1	1.1	7.9	25.6	52.3
Y	LF, HF	-8.7	24.4	50.2	74.3	84.4
Y	HF	-7.2	0.2	26.4	53.2	70.7
X	LF	-6.9	-5.4	-2.2	0.9	4.5
X	HF, LF	-2.2	6.1	21.3	40.8	56.5
X	HF	-0.1	4.1	21.5	39.3	54.6

TABLE 2 BIS: description of dataset and methodology. (HF data: IP indexes, LF: non-IP real **GROSS OUTPUT**)

In the table we display the quantiles of the empirical distributions of the adjusted R^2, denoted $\bar{R}^2$, for different sets of time series regressions.

Panel a)
The regressions in the first three lines involve the **GROSS OUTPUT GROWTH RATES** growth of the **38 non-IP (35 services, Construction, Farms, Forestry-Fishing and related activities)** as dependent variables, while the regressions in the last tree lines involve the growth of the 117 industrial production indexes as dependent variables. The factors used as explanatory variables are estimated from the panel of 38 non-IP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model (**MFFM**) with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the regressions is **1988.Q1-2011.Q4**. In lines 1 and 4 we report the quantiles of $\bar{R}^2$ of the regressions using as explanatory variable the estimated LF factor only. In lines 2 and 5 we report the quantiles of $\bar{R}^2$ of the regressions using as explanatory variables the estimated LF and HF factors. In lines 3 and 6 we report the quantiles of $\bar{R}^2$ of the regressions using as explanatory variable the estimated HF factor only. The regressions in lines 2 and 3 are unrestricted MIDAS regressions. The regressions in lines 4 and 5 allow the estimated coefficients of the LF factor to be different at each quarter.

Panel b)
The regressions in the first three lines involve the **productivity innovations of the 38 non-IP sectors (35 services, Construction, Farms, Forestry-Fishing and related activities) as dependent variables**, while the regressions in the last tree lines involve the **productivity innovations of the 117 industrial production indexes as dependent variables. Note that productivity innovations are computed using the panel of GROSS OUTPUT GROWTH RATES for the LF observables.** The factors used as explanatory variables are estimated from the panel of productivity innovations computed as proposed by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model (**MFFM**) with $K_H = K_L = 1$. **The sample period for the estimation of both the factor model and the regressions is 1988.Q1-2011.Q4, because the productivity shocks can not be computed for the first year of the sample (see Foerster, Sarte, and Watson (2011), especially their equation (B38) on page 10 of their Appendix B.)**

Panel c)
The regressions in the first three lines involve the GROSS OUTPUT growth of the 38 non-IP sectors (35 services, Construction, Farms, Forestry-Fishing and related activities) as dependent variables, while the regressions in the last tree lines involve the growth of the 117 industrial production indexes as dependent variables. The factors used as explanatory variables are estimated from the panel of productivity innovations computed as proposed by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model (**MFFM**) with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the regressions is 1989.Q1-2011.Q4. **Note that productivity innovations are computed using the panel of GROSS OUTPUT GROWTH RATES for the LF observables.**

Panel d)
The regressions in the first three lines involve the GROSS OUTPUT growth of 38 non-IP sectors (35 services, Construction, Farms, Forestry-Fishing and related activities) as dependent variables, while the regressions in the last tree lines involve the growth of the 117 industrial production indexes as dependent variables. The factors used as explanatory variables are estimated from the panel of productivity innovations computed as proposed by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model (**MFFM**) with $K_H = K_L = 1$. Both the contemporaneous and lagged values (only lag 1 is included) of the factors are used as explanatory variables. The choice of including the lags of the factors as regressors, is justified by equations (10) and (12) in Foerster, Sarte, and Watson (2011). The sample period for the estimation of both the factor model and the regressions is 1990.Q1-2011.Q4. **Note that productivity innovations are computed using the panel of GROSS OUTPUT GROWTH RATES for the LF observables.**

Table 3: Adjusted R^2 of the regression of yearly sectoral GDP growth on the HF factor.

Sector	$\bar{R}^2$
Ten sectors with largest $\mathbf{\bar{R}^2}$	
Accommodation	72.06
Truck transportation	60.90
Administrative and support services	56.71
Other transportation and support activities	47.82
Construction	44.03
Other services, except government	42.28
Warehousing and storage	40.99
Misc. professional, scientific, and technical services	40.23
Funds, trusts, and other financial vehicles	38.70
Government enterprises (STATES AND LOCAL)	35.15
Ten sectors with smallest $\mathbf{\bar{R}^2}$	
Insurance carriers and related activities	3.06
Farms	0.35
Forestry, fishing, and related activities	-1.12
General government (STATES AND LOCAL)	-1.12
Federal Reserve banks, credit interm., and rel. activities	-1.54
Water transportation	-3.99
Ambulatory health care services	-4.07
Management of companies and enterprises	-4.24
Hospitals and nursing and residential care facilities	-7.15
Information and data processing services	-9.05

Table 4: Adjusted R^2 of the regression of yearly sectoral GDP growth on the HF and LF factors.

Sector	$\bar{R}^2$
Ten sectors with largest $\mathbf{\bar{R}^2}$	
Construction	73.85
Accommodation	72.62
Administrative and support services	70.69
Truck transportation	60.01
Misc. professional, scientific, and technical services	54.39
Wholesale trade	52.75
Retail trade	52.46
Other services, except government	51.23
Government enterprises (FEDERAL)	50.09
Computer systems design and related services	48.84
Ten sectors with smallest $\mathbf{\bar{R}^2}$	
Broadcasting and telecommunications	6.71
Forestry, fishing, and related activities	6.57
Insurance carriers and related activities	6.15
Securities, commodity contracts, and investments	5.54
Motion picture and sound recording industries	1.14
Information and data processing services	1.01
Ambulatory health care services	-0.65
Federal Reserve banks, credit interm., and rel. activities	-4.65
Water transportation	-7.51
Hospitals and nursing and residential care facilities	-10.82

In the table we display the adjusted R^2, denoted $\bar{R}^2$, for the time series regressions of the growth rates of 42 GDP sectoral indexes on the estimated factors. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The regressions in Table 3 involve a LF explained variable and the estimated HF factor. The regressions in Table 4 involve a LF explained variable and both the HF and LF estimated factors. The regressions in both tables are unrestricted MIDAS regressions.

Table 5: Change in adjusted R^2 of the regression of yearly sectoral GDP growth on the HF factor and the LF factors vs. the regression on the HF factor only.

Sector	change in $\bar{R}^2$
Ten sectors with largest change in $\bar{\mathbf{R}}^2$	
Social assistance	38.89
Computer systems design and related services	37.30
General government (STATES AND LOCAL)	30.67
Construction	29.82
Government enterprises (FEDERAL)	24.52
Rental and leasing services and lessors of intangible assets	23.84
Wholesale trade	22.71
Retail trade	19.41
Management of companies and enterprises	17.10
Real estate	16.34
Ten sectors with smallest change in $\bar{\mathbf{R}}^2$	
Securities, commodity contracts, and investments	-2.20
Pipeline transportation	-2.24
Air transportation	-2.31
Publishing industries (includes software)	-2.67
Broadcasting and telecommunications	-2.97
Waste management and remediation services	-2.97
Federal Reserve banks, credit intermediation, and related activities	-3.11
Motion picture and sound recording industries	-3.22
Water transportation	-3.52
Hospitals and nursing and residential care facilities	-3.68

In the table we display the difference in the adjusted R^2, denoted $\bar{R}^2$, from the regressions of the growth rates of each sectoral GDP index on the HF and LF estimated factors and on the HF factor only. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both factor model and regressions is 1977.Q1-2011.Q4. These regressions are unrestricted MIDAS regressions.

Table 6: Adjusted R^2 of the regression of quarterly industrial production growth on the HF factor.

Sector	$\bar{R}^2$
Ten sectors with largest $\bar{\mathbf{R}}^2$	
Plastics product	73.22
Household and institutional furniture and kitchen cabinet	69.69
Forging and stamping	67.38
Foundries	65.96
Other fabricated metal product	65.87
Coating, engraving, heat treating, and allied activities	65.53
Rubber products ex. Tires	63.24
Machine shops, turned product, and screw, nut, and bolt	61.33
Other Miscellaneous Manufacturing	60.14
Other electrical equipment	58.64
Ten sectors with smallest $\bar{\mathbf{R}}^2$	
Natural gas distribution	-0.26
Animal slaughtering and processing	-0.27
Nonferrous metal (except aluminum) smelting and refining	-0.39
Other Food Except Coffee and Tea	-0.43
Aerospace product and parts	-0.60
Grain and oilseed milling	-0.60
Wineries and Distilleries	-0.67
Dairy product (except frozen)	-0.69
Fruit and vegetable preserving and specialty food	-0.72
Oil and gas extraction	-0.72

Table 7: Adjusted R^2 of the regression of quarterly industrial production growth on the HF and LF factors.

Sector	$\bar{R}^2$
Ten sectors with largest $\bar{\mathbf{R}}^2$	
Plastics product	73.64
Household and institutional furniture and kitchen cabinet	69.40
Forging and stamping	66.72
Coating, engraving, heat treating, and allied activities	66.10
Other fabricated metal product	65.62
Foundries	65.06
Machine shops, turned product, and screw, nut, and bolt	62.26
Rubber products ex. Tires	62.17
Other Miscellaneous Manufacturing	60.94
Other electrical equipment	59.92
Ten sectors with smallest $\bar{\mathbf{R}}^2$	
Wineries and Distilleries	-0.23
Mining and oil and gas field machinery	-0.27
Sugar and confectionery product	-0.42
Coffee and tea	-0.68
Fruit and vegetable preserving and specialty food	-0.88
Other Food Except Coffee and Tea	-1.13
Animal slaughtering and processing	-1.61
Oil and gas extraction	-1.78
Nonferrous metal (except aluminum) smelting and refining	-2.05
Breweries	-2.24

In the table we display the adjusted R^2, denoted $\bar{R}^2$, for the time series regressions of the growth rates of the of 117 industrial production indexes on the estimated factors. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The regressions in Table 6 involve a HF explained variable and the estimated HF factor. The regressions in Table 7 involve a HF explained variable and both the HF and LF estimated factors. As the explanatory variables are observable at high frequency, in order to increase the fit of the model we allow the coefficient of the LF factor to be different in each quarter of the same year.

Table 8: Change in adjusted R^2 of the regression of quarterly industrial production growth on the HF and LF factors vs. the regression on the HF factor only.

Sector	change in $\bar{R}^2$
Ten sectors with largest change in $\bar{\mathbf{R}}^2$	
Computer and peripheral equipment	11.12
Communications equipment	6.57
Grain and oilseed milling	6.50
Newspaper publishers	4.46
Electric power generation, transmission, and distribution	3.95
Railroad rolling stock	3.87
Coal mining	3.50
Periodical, book, and other publishers	3.38
Synthetic dye and pigment	3.01
Dairy product (except frozen)	2.71
Ten sectors with smallest change in $\bar{\mathbf{R}}^2$	
Industrial machinery	-1.73
Coffee and tea	-1.81
Agricultural implement	-1.87
Apparel	-1.88
Pulp mills	-1.88
Engine, turbine, and power transmission equipment	-1.91
Audio and video equipment	-2.19
Petroleum refineries	-2.42
Mining and oil and gas field machinery	-2.60
Breweries	-2.90

In the table we display the difference in the adjusted R^2, denoted $\bar{R}^2$, from the regressions of the growth rates of the 117 industrial production indexes on the HF and LF estimated factors and on the HF factor only. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both factor model and regressions is 1977.Q1-2011.Q4. As the explanatory variables are observable at high frequency, in order to increase the fit of the model we allow the coefficient of the LF factor to be different in each quarter of the same year.

Table 9: Adjusted R^2 of selected indexes on the estimated **1 HF and 1 LF factors** (HF data: IP indexes, LF: non-IP real **value added GDP**).

Sector	(1) $\bar{R}^2(HF)$	(2) $\bar{R}^2(LF)$	(3) $\bar{R}^2(HF+LF)$	(3) - (1)
PANEL a)				
REGRESSORS: factors extracted from sectoral output growth (X and Y)				
HF observations				
Industrial Production	89.46	-0.08	90.03	0.57
LF observations				
GDP	60.39	20.22	85.48	25.09
GDP - Manifacturing	74.20	-0.76	75.89	1.69
GDP - Agriculture, forestry, fishing, and hunting	-0.61	4.85	4.88	5.49
GDP - Construction	44.03	24.88	73.85	29.82
GDP - Wholesale trade	30.04	19.06	52.75	22.71
GDP - Retail trade	33.05	16.06	52.46	19.41
GDP - Transportation and warehousing	54.55	-1.43	54.81	0.26
GDP - Information	18.12	-2.54	15.85	-2.26
GDP - Finance, insurance, real estate, rental, and leasing	6.65	21.82	31.69	25.04
GDP - Professional and business services	47.29	19.17	70.74	23.45
GDP - Educational services, health care, and social assistance	-10.52	-2.96	-14.25	-3.74
GDP - Arts, entert., recreation, accommod., and food services	63.10	1.14	66.57	3.47
GDP - Government	-2.03	12.38	11.98	14.00
PANEL b)				
REGRESSORS: contemporaneous values of factors extracted from innovations to sectoral productivity (ε_t in Foerster, Sarte, and Watson (2011)).				
HF observations				
Industrial Production	69.30	6.08	75.95	6.65
LF observations				
GDP	29.24	31.63	66.45	37.21
GDP - Manifacturing	53.31	10.41	67.13	13.82
GDP - Agriculture, forestry, fishing, and hunting	3.99	-1.48	2.47	-1.52
GDP - Construction	20.22	30.65	56.00	35.78
GDP - Wholesale trade	15.79	36.68	58.44	42.65
GDP - Retail trade	13.55	54.86	76.81	63.26
GDP - Transportation and warehousing	42.09	-0.34	43.18	1.09
GDP - Information	13.60	1.05	15.24	1.64
GDP - Finance, insurance, real estate, rental, and leasing	7.83	4.63	13.37	5.54
GDP - Professional and business services	31.82	24.78	61.16	29.35
GDP - Educational services, health care, and social assistance	-4.95	10.15	6.55	11.50
GDP - Arts, entert., recreation, accommod., and food services	36.21	30.30	72.27	36.06
GDP - Government	4.23	1.02	5.56	1.34

TABLE 9: Adjusted R^2 of selected indexes on the estimated **1 HF and 1 LF factors, and their lagged values** (HF data: IP indexes, LF: non-IP real **value added GDP**).

Sector	(1) $\bar{R}^2(HF)$	(2) $\bar{R}^2(LF)$	(3) $\bar{R}^2(HF+LF)$	(3) - (1)
PANEL c)				
REGRESSORS: factors extracted from innovations to sectoral productivity				
(ε_t in Foerster, Sarte, and Watson (2011)) both contemporaneous and lagged values (only first lag).				
HF observations				
Industrial Production	76.77	2.10	82.91	6.15
LF observations				
GDP	38.30	32.43	70.56	32.26
GDP - Manifacturing	62.49	6.75	69.26	6.77
GDP - Agriculture, forestry, fishing, and hunting	23.81	-3.63	18.68	-5.13
GDP - Construction	28.67	38.78	63.32	34.64
GDP - Wholesale trade	16.53	37.27	55.03	38.50
GDP - Retail trade	16.14	55.02	73.00	56.86
GDP - Transportation and warehousing	54.82	-3.94	53.32	-1.50
GDP - Information	34.39	13.36	35.75	1.36
GDP - Finance, insurance, real estate, rental, and leasing	-0.97	9.11	1.43	2.40
GDP - Professional and business services	33.43	41.52	68.75	35.32
GDP - Educational services, health care, and social assistance	-4.43	26.08	10.60	15.03
GDP - Arts, entert., recreation, accommod., and food services	35.48	25.02	74.47	38.99
GDP - Government	4.33	1.25	13.85	9.53

TABLE 9: description of dataset and methodology (HF data: IP indexes, LF: non-IP real **value added GDP**)

In the table we display the adjusted R^2, denoted $\bar{R}^2$, of the regression of growth rates of selected HF and LF indexes on the HF factor (column $\bar{R}^2(HF)$), the LF factor (column $\bar{R}^2(LF)$) and both the HF and LF factors (column $\bar{R}^2(LF+HF)$). The last column displays the difference of the values in column $\bar{R}^2(LF+HF)$ and column $\bar{R}^2(HF)$, i.e. the increment in the adjusted R^2 when the LF factor is added as a regressor to the HF factor.

PANEL a)

The GDP indexes used in this table are aggregates of the indexes used to estimate the factors. The factors are estimated from the panel of 42 non-IP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the regressions is 1977.Q1-2011.Q4.

PANEL b)

The GDP indexes are the same as in Panel a). The factors used as explanatory variables are estimated from the panel of productivity innovations computed as proposed by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model (**MFFM**) with $K_H = K_L = 1$. **The sample period for the estimation of both the factor model and the regressions is 1978.Q1-2011.Q4, because the productivity shocks can not be computed for the first year of the sample (see Foerster, Sarte, and Watson (2011), especially their equation (B38) on page 10 of their Appendix B.)**

PANEL c)

The GDP indexes are the same as in Panel a). The factors used as explanatory variables are estimated from the panel of productivity innovations computed as proposed by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model (**MFFM**) with $K_H = K_L = 1$. **Both the contemporaneous and lagged values (only lag 1 is included) of the factors are used as explanatory variables. The choice of including the lags of the factors as regressors, is justified by equations (10) and (12) in Foerster, Sarte, and Watson (2011)**. The sample period for the estimation of both the factor model and the regressions is 1979.Q1-2011.Q4.

TABLE 9 BIS: Adjusted R^2 of selected indexes on the estimated 1 HF and 1 LF factors. (HF data: IP indexes, LF: non-IP real **GROSS OUTPUT**)

Sector	(1) $\bar{R}^2(HF)$	(2) $\bar{R}^2(LF)$	(3) $\bar{R}^2(HF+LF)$	(3) - (1)
PANEL a)				
REGRESSORS: factors extracted from sectoral output growth (X and Y)				
HF observations				
Industrial Production	89.28	-0.22	90.01	0.73
LF observations				
GO (all sectors)	70.75	16.73	94.89	24.14
GO - Manifacturing	90.22	-0.38	94.68	4.47
GO - Agriculture, forestry, fishing, and hunting	-8.85	0.12	-9.13	-0.28
GO - Construction	28.13	29.35	65.29	37.16
GO - Wholesale trade	85.18	-3.56	85.52	0.34
GO - Retail trade	81.68	-2.75	82.83	1.15
GO - Transportation and warehousing	75.42	3.54	83.81	8.39
GO - Information	25.46	28.87	61.91	36.46
GO - Finance, insurance, real estate, rental, and leasing	17.72	22.89	46.49	28.78
GO - Professional and business services	45.12	33.85	88.60	43.48
GO - Educational services, health care, and social assistance	3.05	3.28	7.15	4.10
GO - Arts, entertainment, recreation, accommodation, and food services	71.14	0.38	75.42	4.28
GO - Government	12.24	-0.52	12.32	0.08
PANEL b)				
REGRESSORS: contemporaneous values of factors extracted from innovations to sectoral productivity (ε_t in Foerster, Sarte, and Watson (2011)).				
HF observations				
Industrial Production	70.52	9.18	80.60	10.08
LF observations				
GO (all sectors)	48.57	27.61	85.09	36.51
GO - Manifacturing	70.00	16.02	93.50	23.50
GO - Agriculture, forestry, fishing, and hunting	-11.65	-3.17	-16.24	-4.58
GO - Construction	19.12	20.95	45.86	26.74
GO - Wholesale trade	78.27	6.95	91.45	13.18
GO - Retail trade	73.40	4.97	83.86	10.46
GO - Transportation and warehousing	68.17	7.51	81.14	12.97
GO - Information	4.57	59.81	78.09	73.51
GO - Finance, insurance, real estate, rental, and leasing	6.60	13.04	22.88	16.28
GO - Professional and business services	37.74	36.51	84.23	46.49
GO - Educational services, health care, and social assistance	12.33	2.55	16.10	3.76
GO - Arts, entert., recreation, accommod., and food services	66.27	-0.54	69.36	3.09
GO - Government	13.12	-1.39	11.95	-1.16

TABLE 9 BIS: Adjusted R^2 of selected indexes on the estimated 1 HF and 1 LF factors, and their lagged values. (HF data: IP indexes, LF: non-IP real **GROSS OUTPUT**)

Sector	(1) $\bar{R}^2(HF)$	(2) $\bar{R}^2(LF)$	(3) $\bar{R}^2(HF+LF)$	(3) - (1)
PANEL c)				
REGRESSORS: factors extracted from innovations to sectoral productivity				
(ε_t in Foerster, Sarte, and Watson (2011)) both contemporaneous and lagged values (only first lag).				
HF observations				
Industrial Production	71.41	5.77	80.70	9.29
LF observations				
GO (all sectors)	53.15	24.71	84.74	31.59
GO - Manifacturing	74.63	13.09	92.69	18.06
GO - Agriculture, forestry, fishing, and hunting	-40.11	-7.36	-50.94	-10.83
GO - Construction	6.96	17.98	26.30	19.34
GO - Wholesale trade	81.76	2.47	91.74	9.97
GO - Retail trade	76.44	2.86	82.66	6.22
GO - Transportation and warehousing	84.37	15.69	88.58	4.20
GO - Information	7.93	64.94	95.04	87.11
GO - Finance, insurance, real estate, rental, and leasing	14.64	13.83	28.03	13.39
GO - Professional and business services	41.01	41.96	82.76	41.75
GO - Educational services, health care, and social assistance	-4.03	3.42	-0.76	3.27
GO - Arts, entert., recreation, accommod., and food services	74.56	-3.75	71.97	-2.59
GO - Government	75.69	14.80	78.36	2.66

TABLE 9 BIS: description of dataset and methodology. (HF data: IP indexes, LF: non-IP real **GROSS OUTPUT**)

In the table we display the adjusted R^2, denoted $\bar{R}^2$, of the regression of growth rates of selected HF and LF indexes on the HF factor (column $\bar{R}^2(HF)$), the LF factor (column $\bar{R}^2(LF)$) and both the HF and LF factors (column $\bar{R}^2(LF+HF)$). The last column displays the difference of the values in column $\bar{R}^2(LF+HF)$ and column $\bar{R}^2(HF)$, i.e. the increment in the adjusted R^2 when the LF factor is added as a regressor to the HF factor.

PANEL a)
The GDP indexes used in this table are aggregates of the indexes used to estimate the factors. The factors are estimated from the panel of **38 non-IP sectors** and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the regressions is **1988.Q1-2011.Q4**.

PANEL b)
The GROSS OUTPUT indexes are the same as in Panel a). The factors used as explanatory variables are estimated from the panel of productivity innovations computed as proposed by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model (**MFFM**) with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the regressions is 1989.Q1-2011.Q4, because the productivity shocks can not be computed for the first year of the sample. **Productivity innovations are computed using the panel of GROSS OUTPUT GROWTH RATES for the LF observables.**

PANEL c)
Corresponds to PANEL c) in Table 9. The sample period for the estimation of both the factor model and the regressions is 1990.Q1-2011.Q4. **Note that productivity innovations are computed using the panel of GROSS OUTPUT GROWTH RATES for the LF observables.**

Table 10: Correlation matrix of the estimated HF and LF factors.

	$\hat{f}_{1,t}$	$\hat{f}_{2,t}$	$\hat{f}_{3,t}$	$\hat{f}_{4,t}$	$\hat{g}_t$
$\hat{f}_{1,t}$	1.000	0.663	0.254	0.141	0.000
$\hat{f}_{2,t}$	0.663	1.000	0.668	0.148	0.000
$\hat{f}_{3,t}$	0.254	0.668	1.000	0.639	0.000
$\hat{f}_{4,t}$	0.141	0.148	0.639	1.000	0.000
$\hat{g}_t$	0.000	0.000	0.000	0.000	1.000

In the table we display the correlation matrix of the stacked vector of estimated factors $(\hat{f}_{1,t}, \hat{f}_{2,t}, \hat{f}_{3,t}, \hat{f}_{4,t}, \hat{g}_t)$. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the regressions is 1977.Q1-2011.Q4.

Table 11: Regressions of observed HF and LF observables on estimated factors: quantiles of $\hat{\lambda}_i$ and $\hat{b}_i$.

	Quantile				
Coeff.	10%	25%	50%	75%	90%
$\hat{\lambda}_i$	0.0670	0.2428	0.5116	0.6200	0.7546
$\hat{b}_i$	-0.2474	0.0176	0.2058	0.3664	0.4856

In the table we display the quantiles of the empirical distributions of the estimated loadings $\hat{\lambda}_i$ and $\hat{b}_i$ of the HF and LF factors, i.e. the elements of the estimated vectors $\hat{\Lambda}$ and $\hat{B}$, respectively. The factors and the loadings are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the regressions is 1977.Q1-2011.Q4.

Table 12: Estimates of the unconstrained reduced-form VAR (1) model for the factor process.

We estimate the following unconstrained reduced-form VAR(1) on the factor process:

$$\begin{bmatrix} f_{1,t} \\ f_{2,t} \\ f_{3,t} \\ f_{4,t} \\ g_t \end{bmatrix} = a + A \begin{bmatrix} f_{1,t-1} \\ f_{2,t-1} \\ f_{3,t-1} \\ f_{4,t-1} \\ g_{t-1} \end{bmatrix} + \zeta_t, \qquad \zeta_t \sim N(0, \Sigma_\zeta). \tag{T.1}$$

The estimates are given by:

$$\hat{A} = \begin{bmatrix} \mathbf{-0.45} & 0.35 & -0.06 & \mathbf{0.82} & -0.09 \\ \mathbf{(0.16)} & (0.23) & (0.39) & \mathbf{(0.17)} & (0.11) \\ -0.36 & -0.03 & 0.47 & 0.27 & -0.30 \\ (0.23) & (0.34) & (0.57) & (0.25) & (0.16) \\ -0.17 & -0.03 & 0.22 & -0.06 & -0.10 \\ (0.17) & (0.25) & (0.42) & (0.19) & (0.12) \\ -0.30 & 0.33 & 0.14 & -0.09 & 0.08 \\ (0.28) & (0.41) & (0.68) & (0.30) & (0.20) \\ 0.16 & 0.29 & -0.18 & 0.22 & \mathbf{0.36} \\ (0.23) & (0.33) & (0.55) & (0.24) & \mathbf{(0.16)} \end{bmatrix},$$

$$\hat{\Sigma}_\zeta = \begin{bmatrix} 0.3444 & 0.2492 & 0.0986 & 0.0796 & -0.1096 \\ (0.0591) & (0.0000) & (0.0000) & (0.0000) & (0.0000) \\ 0.2492 & 0.7319 & 0.3615 & 0.1043 & 0.1545 \\ (0.0680) & (0.1255) & (0.0000) & (0.0000) & (0.0000) \\ 0.0986 & 0.3615 & 0.3981 & 0.4386 & 0.1116 \\ (0.0465) & (0.0788) & (0.0683) & (0.0000) & (0.0000) \\ 0.0796 & 0.1043 & 0.4386 & 1.0657 & -0.0434 \\ (0.0741) & (0.1078) & (0.0952) & (0.1828) & (0.0000) \\ -0.1096 & 0.1545 & 0.1116 & -0.0434 & 0.6865 \\ (0.0604) & (0.0880) & (0.0648) & (0.1039) & (0.1177) \end{bmatrix}.$$

The correlation matrix corresponding to the estimated variance-covariance matrix $\hat{\Sigma}_\zeta$ is:

$$\begin{bmatrix} 1.0000 & 0.4964 & 0.2664 & 0.1315 & -0.2254 \\ 0.4964 & 1.0000 & 0.6697 & 0.1181 & 0.2180 \\ 0.2664 & 0.6697 & 1.0000 & 0.6733 & 0.2135 \\ 0.1315 & 0.1181 & 0.6733 & 1.0000 & -0.0507 \\ -0.2254 & 0.2180 & 0.2135 & -0.0507 & 1.0000 \end{bmatrix}.$$

The estimated values of the constant vector $\hat{a}$ are not reported because they are not significantly different from zero at the 5% level. The VAR (1) model is estimated by OLS equation by equation. Significant estimates at 5% level are displayed in bold and standard errors are reported in parentheses. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the VAR (1) model is 1977.Q1-2011.Q4.

Table 13: Estimates of the constrained reduced-form VAR (1) model for the factor process.

We estimate the following constrained reduced-form VAR(1) on the factor process:

$$\begin{bmatrix} f_{1,t} \\ f_{2,t} \\ f_{3,t} \\ f_{4,t} \\ g_t \end{bmatrix} = A \begin{bmatrix} f_{1,t-1} \\ f_{2,t-1} \\ f_{3,t-1} \\ f_{4,t-1} \\ g_{t-1} \end{bmatrix} + \zeta_t, \qquad \zeta_t \sim N(0,\Sigma) \, , \tag{T.2}$$

where:

$$A = \begin{bmatrix} 0 & 0 & 0 & r_H & a \\ 0 & 0 & 0 & r_H^2 & a(1+r_H) \\ 0 & 0 & 0 & r_H^3 & a(1+r_H+r_H^2) \\ 0 & 0 & 0 & r_H^4 & a(1+r_H+r_H^2+r_H^3) \\ m_1 & m_2 & m_3 & m_4 & r_L \end{bmatrix}, \tag{T.3}$$

and

$$\Sigma_\zeta = \begin{bmatrix} \sigma_H^2 & & & & \\ \sigma_H^2 r_H & \sigma_H^2(1+r_H^2) & & & \\ \sigma_H^2 r_H^2 & \sigma_H^2 r_H(1+r_H^2) & \sigma_H^2(1+r_H^2+r_H^4) & & \\ \sigma_H^2 r_H^3 & \sigma_H^2 r_H^2(1+r_H^2) & \sigma_H^2 r_H(1+r_H^2+r_H^4) & \sigma_H^2(1+r_H^2+r_H^4+r_H^6) & \\ \rho_{HL}\sigma_H\sigma_L & (1+r)\rho_{HL}\sigma_H\sigma_L & \rho_{HL}\sigma_H\sigma_L(1+r+r^2) & \rho_{HL}\sigma_H\sigma_L(1+r+r^2+r^3) & \sigma_L^2 \end{bmatrix}. \tag{T.4}$$

The estimates are given by:

$$\hat{A} = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.6542 & -0.0268 \\ 0.0000 & 0.0000 & 0.0000 & 0.4280 & -0.0443 \\ 0.0000 & 0.0000 & 0.0000 & 0.2800 & -0.0557 \\ 0.0000 & 0.0000 & 0.0000 & 0.1832 & -0.0632 \\ 0.1677 & 0.2821 & -0.1756 & 0.2207 & 0.3643 \end{bmatrix}, \ \hat{\Sigma}_\zeta = \begin{bmatrix} 0.5623 & 0.3678 & 0.2406 & 0.1574 & 0.0033 \\ 0.3678 & 0.8029 & 0.5252 & 0.3436 & 0.0055 \\ 0.2406 & 0.5252 & 0.9059 & 0.5926 & 0.0070 \\ 0.1574 & 0.3436 & 0.5926 & 0.9499 & 0.0079 \\ 0.0033 & 0.0055 & 0.0070 & 0.0079 & 0.6664 \end{bmatrix},$$

Coefficient	Estimate	St. Error
r_H	**0.6542**	**0.0651**
a	-0.0268	0.0665
m_1	0.1677	0.2134
m_2	0.2821	0.3008
m_3	-0.1756	0.4968
m_4	0.2207	0.2233
ϕ	**0.3643**	**0.1438**
σ_H	**0.7498**	**0.1283**
σ_L	**0.8163**	**0.2743**
ρ_{HL}	0.0055	0.0962

The VAR (1) model is estimated by Maximum Likelihood. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the VAR (1) model is 1977.Q1-2011.Q4.

Table 14: Adjusted R^2 of the regression of yearly sectoral GDP growth on the HF factor.

Sector	$\bar{R}^2$
Accommodation	72.06
Truck transportation	60.90
Administrative and support services	56.71
Other transportation and support activities	47.82
Construction	44.03
Other services, except government	42.28
Warehousing and storage	40.99
Miscellaneous professional, scientific, and technical services	40.23
Funds, trusts, and other financial vehicles	38.70
Government enterprises (STATES AND LOCAL)	35.15
Legal services	33.25
Retail trade	33.05
Wholesale trade	30.04
Air transportation	27.25
Food services and drinking places	27.13
Government enterprises (FEDERAL)	25.57
Performing arts, spectator sports, museums, and related activities	22.43
Publishing industries (includes software)	21.69
Amusements, gambling, and recreation industries	19.53
Real estate	19.38
Rail transportation	18.90
Waste management and remediation services	12.73
Pipeline transportation	11.90
Computer systems design and related services	11.54
Educational services	10.49
Broadcasting and telecommunications	9.68
Securities, commodity contracts, and investments	7.74
Social assistance	6.33
Rental and leasing services and lessors of intangible assets	6.16
Motion picture and sound recording industries	4.35
Transit and ground passenger transportation	4.02
General government (FEDERAL)	3.94
Insurance carriers and related activities	3.06
Farms	0.35
Forestry, fishing, and related activities	-1.12
General government (STATES AND LOCAL)	-1.12
Federal Reserve banks, credit intermediation, and related activities	-1.54
Water transportation	-3.99
Ambulatory health care services	-4.07
Management of companies and enterprises	-4.24
Hospitals and nursing and residential care facilities	-7.15
Information and data processing services	-9.05

In the table we display the adjusted R^2, denoted $\bar{R}^2$, for the time series regressions of each of the of 42 GDP sectors on the estimated HF factor. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both factor model and regressions is 1977.Q1-2011.Q4. The regressions in this table are unrestricted MIDAS regressions.

Table 15: Adjusted R^2 of the regression of yearly sectoral GDP growth on the HF and LF factors.

Sector	$\bar{R}^2$
Construction	73.85
Accommodation	72.62
Administrative and support services	70.69
Truck transportation	60.01
Miscellaneous professional, scientific, and technical services	54.39
Wholesale trade	52.75
Retail trade	52.46
Other services, except government	51.23
Government enterprises (FEDERAL)	50.09
Computer systems design and related services	48.84
Other transportation and support activities	46.02
Social assistance	45.21
Warehousing and storage	44.90
Funds, trusts, and other financial vehicles	44.86
Legal services	44.49
Government enterprises (STATES AND LOCAL)	41.52
Real estate	35.72
Food services and drinking places	35.51
Rental and leasing services and lessors of intangible assets	30.00
General government (STATES AND LOCAL)	29.55
Air transportation	24.94
Performing arts, spectator sports, museums, and related activities	24.11
Rail transportation	20.19
Publishing industries (includes software)	19.02
Amusements, gambling, and recreation industries	18.23
Educational services	13.71
Transit and ground passenger transportation	13.04
Management of companies and enterprises	12.87
General government (FEDERAL)	11.74
Waste management and remediation services	9.76
Pipeline transportation	9.66
Farms	8.70
Broadcasting and telecommunications	6.71
Forestry, fishing, and related activities	6.57
Insurance carriers and related activities	6.15
Securities, commodity contracts, and investments	5.54
Motion picture and sound recording industries	1.14
Information and data processing services	1.01
Ambulatory health care services	-0.65
Federal Reserve banks, credit intermediation, and related activities	-4.65
Water transportation	-7.51
Hospitals and nursing and residential care facilities	-10.82

In the table we display the adjusted R^2, denoted $\bar{R}^2$, for the time series regressions of each of the of 42 GDP sectors on the estimated HF and LF factors. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both factor model and regressions is 1977.Q1-2011.Q4. The regressions in this table are unrestricted MIDAS regressions.

Table 16: Change in adjusted R^2 of the regression of yearly sectoral GDP growth on the HF factor and the LF factors vs. the regression on the HF factor only.

Sector	change in $\bar{R}^2$	$\hat{B}$
Social assistance	38.89	0.59
Computer systems design and related services	37.30	0.58
General government (STATES AND LOCAL)	30.67	0.53
Construction	29.82	0.51
Government enterprises (FEDERAL)	24.52	0.47
Rental and leasing services and lessors of intangible assets	23.84	0.47
Wholesale trade	22.71	0.46
Retail trade	19.41	0.42
Management of companies and enterprises	17.10	0.41
Real estate	16.34	0.40
Miscellaneous professional, scientific, and technical services	14.15	0.37
Administrative and support services	13.97	0.36
Legal services	11.25	0.34
Information and data processing services	10.06	-0.34
Transit and ground passenger transportation	9.02	0.32
Other services, except government	8.95	0.30
Food services and drinking places	8.38	0.30
Farms	8.35	0.31
General government (FEDERAL)	7.80	-0.30
Forestry, fishing, and related activities	7.69	-0.30
Government enterprises (STATES AND LOCAL)	6.37	0.27
Funds, trusts, and other financial vehicles	6.16	-0.26
Warehousing and storage	3.90	0.22
Ambulatory health care services	3.42	-0.24
Educational services	3.21	0.23
Insurance carriers and related activities	3.09	0.23
Performing arts, spectator sports, museums, and related activities	1.68	0.19
Rail transportation	1.29	0.18
Accommodation	0.56	-0.11
Truck transportation	-0.89	0.06
Amusements, gambling, and recreation industries	-1.30	0.11
Other transportation and support activities	-1.80	-0.00
Securities, commodity contracts, and investments	-2.20	0.09
Pipeline transportation	-2.24	-0.08
Air transportation	-2.31	0.04
Publishing industries (includes software)	-2.67	0.02
Broadcasting and telecommunications	-2.97	0.03
Waste management and remediation services	-2.97	0.02
Federal Reserve banks, credit intermediation, and related activities	-3.11	0.06
Motion picture and sound recording industries	-3.22	0.03
Water transportation	-3.52	0.02
Hospitals and nursing and residential care facilities	-3.68	-0.01

In the table we display the difference in the adjusted R^2 ($\bar{R}^2$) from the regressions of each industrial production index growth on the HF and LF estimated factors and on the HF factor only. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both factor model and regressions is 1977.Q1-2011.Q4. The regressions in this table are unrestricted MIDAS regressions.

Table 17: Simulation results for DGP with 1 HF and 1 LF factors and different loading Δ_j.

					R^2 quantiles				
Factor	N_H	N_L	T_H	T_L	5%	25%	50%	75%	95%
DESIGN 1: $\Delta_j = \hat{\Delta}_j$									
HF	117	45	140	35	**95**	**96**	**97**	**97**	**98**
HF	498	180	560	140	**99**	**99**	**99**	**99**	**99**
LF	117	45	140	35	**34**	**55**	**65**	**74**	**82**
LF	498	180	560	140	**86**	**89**	**92**	**93**	**95**
DESIGN 2: $\Delta_j = 2 \cdot \hat{\Delta}_j$									
HF	117	45	140	35	**95**	**96**	**97**	**97**	**98**
HF	498	180	560	140	**99**	**99**	**99**	**99**	**99**
LF	117	45	140	35	**31**	**52**	**63**	**72**	**81**
LF	498	180	560	140	**83**	**87**	**90**	**92**	**94**
DESIGN 3: $\Delta_j = 5 \cdot \hat{\Delta}_j$									
HF	117	45	140	35	**59**	**84**	**91**	**95**	**97**
HF	498	180	560	140	**90**	**95**	**96**	**98**	**99**
LF	117	45	140	35	**3**	**16**	**31**	**46**	**65**
LF	498	180	560	140	**41**	**57**	**68**	**77**	**87**

We consider three simulation designs for the mixed frequency factor model in equation (1), in the case of 4 HF subperiods, and equations (T.2) - (T.4) in table 13, and we assume that the numbers of factors are $K_{LF} = K_{HF} = 1$ both for simulation and in estimation. The number of simulations for each design is 5000. The mixed frequency panels of observations are simulated using the values of the parameters reported in the following table:

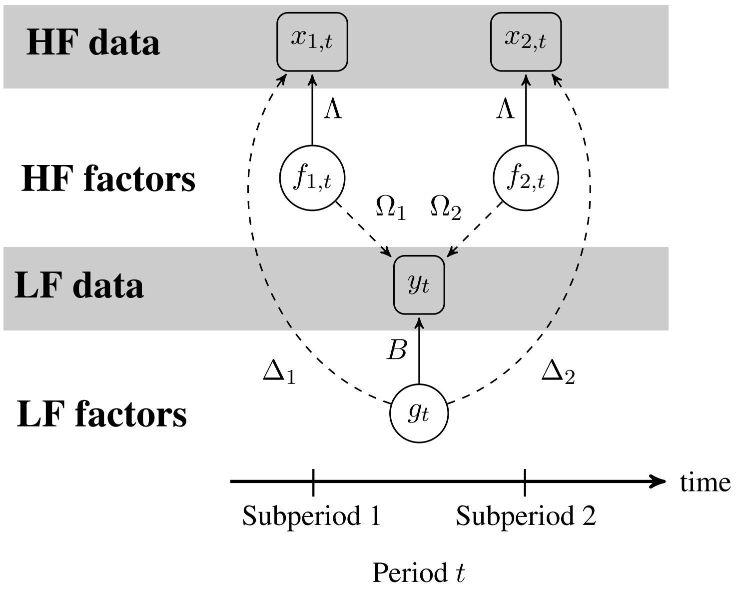
Param.	value	Param.	mean	std.dev.	Param.	mean	std.dev.
r_H	0.6542	Δ_1	-0.0021	0.1610	σ_ε	0.8909	0.1389
a	0.0000	Δ_2	0.0197	0.1557	σ_u	0.7726	0.1343
m_1	0.0000	Δ_3	-0.0012	0.1450			
m_2	0.0000	Δ_4	0.0040	0.1463			
m_3	0.0000	B	-0.1735	0.2547			
m_4	0.0000	Ω_1	0.1986	0.2506			
ϕ	0.3643	Ω_2	0.1311	0.2874			
σ_H	0.7498	Ω_3	-0.1798	0.4990			
σ_L	0.8163	Ω_4	0.1302	0.2258			
ρ_{HL}	0.0000	Λ	0.4378	0.2528			

All the simulated loadings, with the exception of Δ_j, are drawn from independent normal distributions, with mean and variance equal to the corresponding sample moments of the estimated loadings from our macro dataset, reported in the previous table. Design 1 maintains the same distributions as in our macro dataset to simulate the loadings Δ_j, while Design 2 (resp. Design 3) is such that the simulated values of the Δ_j loadings are 2 (resp. 5) times bigger than in our macro-dataset. The variance-covariance matrices of the simulated innovations are diagonal, and their diagonal elements are bootstrapped from the values in the diagonals of the estimated variance-covariance matrices in our macro dataset. The averages and standard deviations of the square roots of the diagonal elements of these estimated matrices are reported in the table, on the lines named σ_ε and σ_u, respectively. For each simulation design we report one table displaying:

- Line 1: the quantiles of the R^2 of the regression of the true HF factor on HF factor estimated from simulated panels with same TS and CS dimensions as in our macro-dataset;
- Line 2: the quantiles of the R^2 of the regression of the true HF factor on HF factor estimated from simulated panels such that both the CS and TS dimensions are four times larger than in our macro-dataset;
- Line 3: the quantiles of the R^2 of the regression of the true LF factor on LF factor estimated from simulated panels with same TS and CS dimensions as in our macro-dataset;
- Line 4: the quantiles of the R^2 of the regression of the true LF factor LF factor estimated from simulated panels such that both the CS and TS dimensions are four times larger than in our macro-dataset.

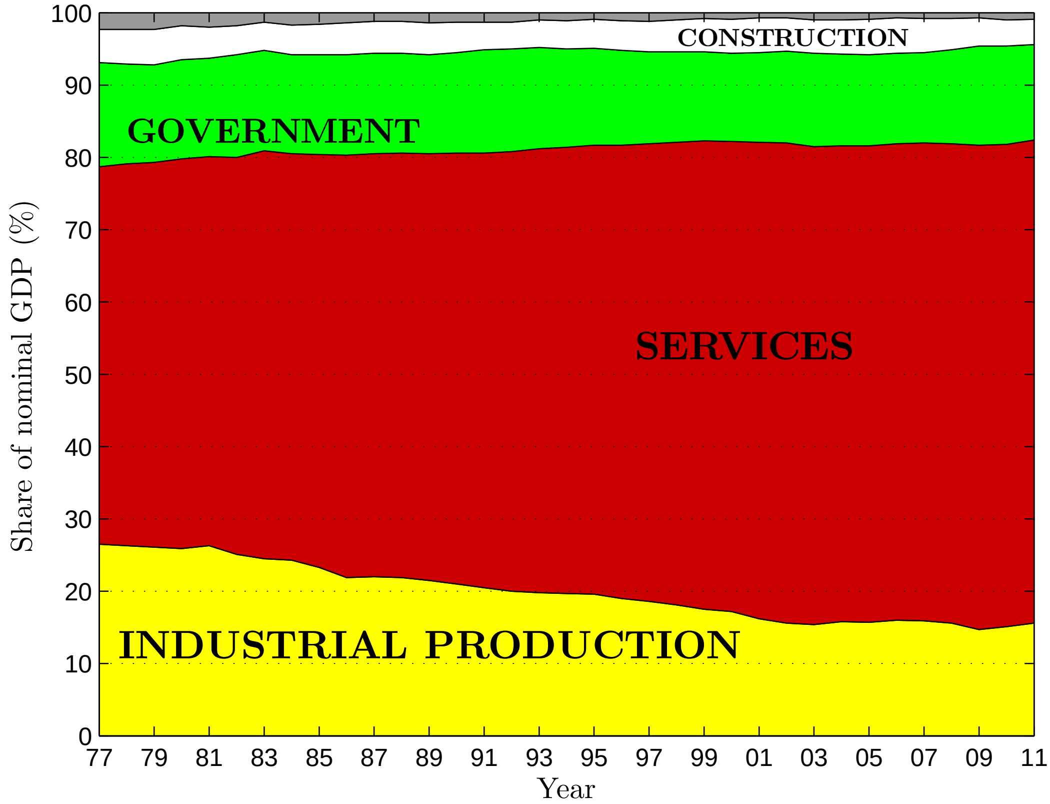
FIGURES

Figure 1: The model structure in the case of two high frequency subperiods.



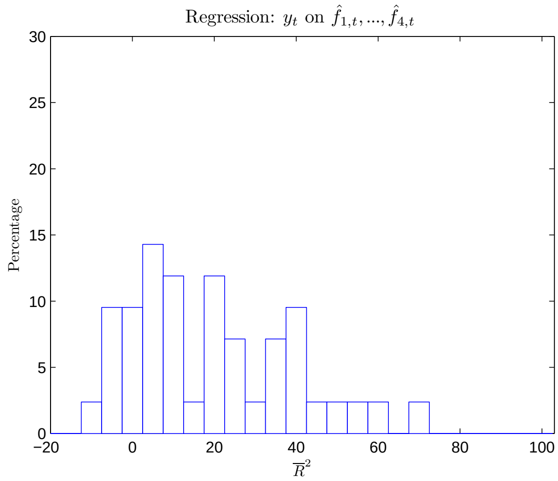
The Figure displays the schematic representation of the mixed-frequency factor model described in Section 2.1.

Figure 2: Evolution of sectoral decomposition of US nominal GDP.

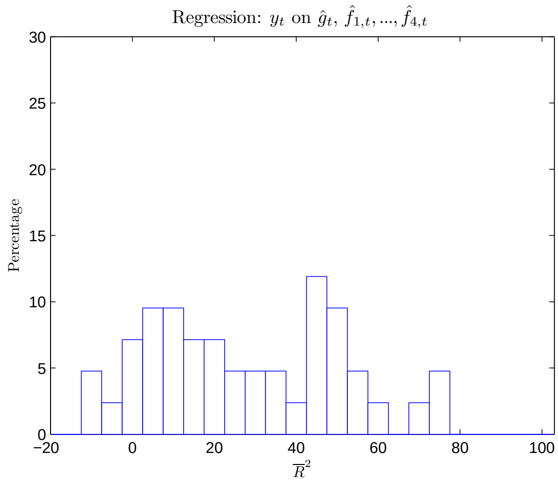


The Figure displays the evolution from 1977 to 2011 of the sectoral decomposition of US nominal GDP. We aggregate the shares of different sectors available from the website of the US Bureau of Economic Analysis, according to their NAICS codes, in 5 different *macro* sectors: Industrial Production (yellow), Services (red), Government (green), Construction (white), Others (grey).

Figure 3: Adjusted R^2 of the regression of yearly sectoral GDP growth on estimated factors.



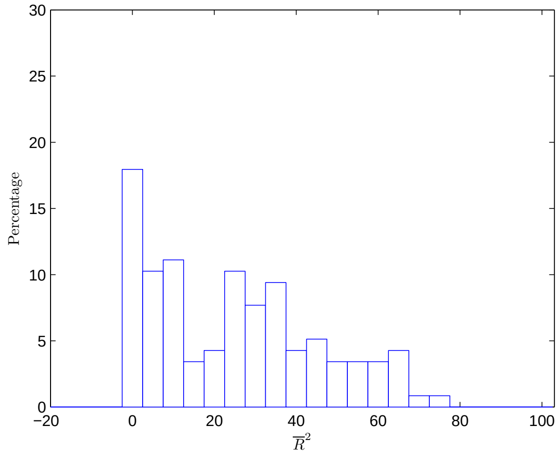
(a) Adjusted R^2 of the regression of yearly sectoral GDP growth on the HF factor.



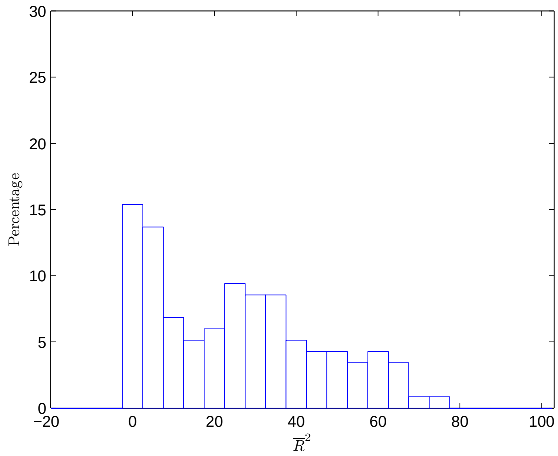
(b) Adjusted R^2 of the regression of yearly sectoral GDP growth on the HF and LF factors.

In Panel (a) we show the histogram of the adjusted R^2, denoted $\bar{R}^2$, of the regressions of the yearly growth rates of sectoral GDP indexes on the estimated HF factor. In Panel (b) we show the histogram of the adjusted R^2 of the regressions of the same growth rates on the estimated HF and LF factors. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the regressions is 1977.Q1-2011.Q4.

Figure 4: Adjusted R^2 of the regression of quarterly industrial production growth on estimated factors.



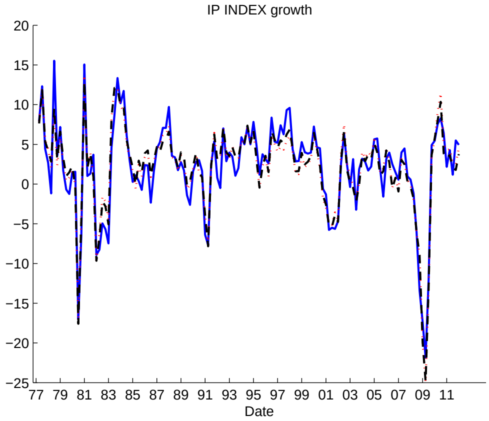
(a) Adjusted R^2 of the regression of quarterly industrial production growth on the HF factor.



(b) Adjusted R^2 of the regression of quarterly industrial production growth on the HF and LF factors.

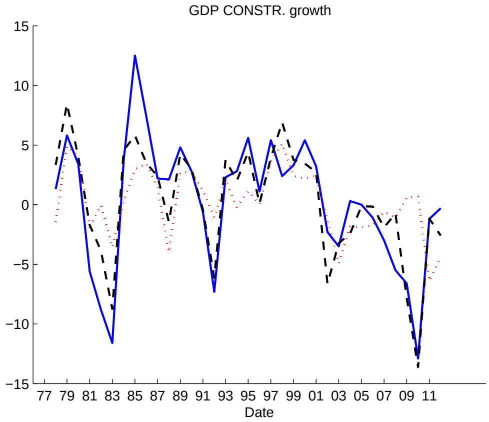
In Panel (a) we show the histogram of the adjusted R^2, denoted $\bar{R}^2$, of the regressions of the quarterly growth rates of the industrial production indexes on the estimated HF factor. In Panel (b) we show the histogram of the adjusted R^2 of the regressions of the same growth rates on the estimated HF and LF factors. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the regressions is 1977.Q1-2011.Q4.

Figure 5: Regression of LF and HF indexes on estimated factors.

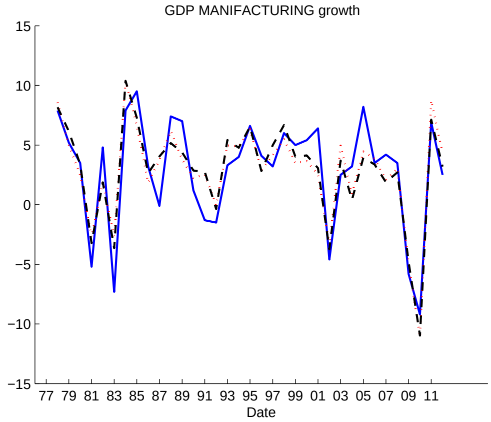


(a) HF Index: Industrial Production Index growth.

(b) LF Index: Aggregate GDP Index growth.



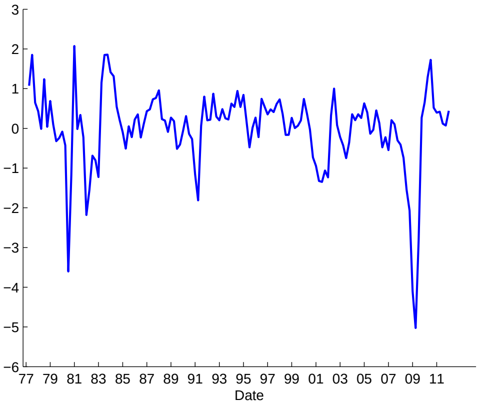
(c) LF Index: GDP-Construction Index growth.



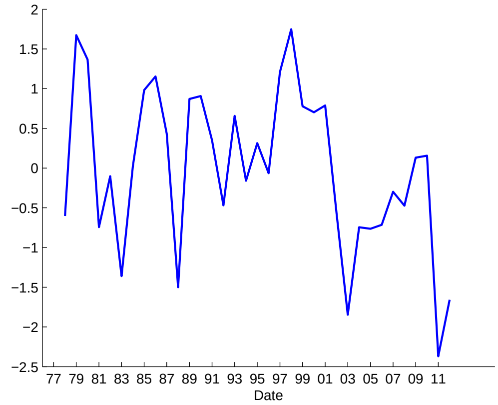
(d) LF Index: GDP-Manifacturing Index growth.

Each panel displays the time series of the growth rate a certain HF or LF index (solid line), its fitted value obtained from a regression of the index on the HF factor (dotted line), and its fitted value obtained from a regression of the index on both the HF and LF factors (dashed line). The first three indexes reported in the panels are aggregates of the indexes used to estimate the factors. The fourth index (GDP-Manufacturing) is constructed from sub-indexes not used for the estimation of the factors. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of both the factor model and the regressions is 1977.Q1-2011.Q4.

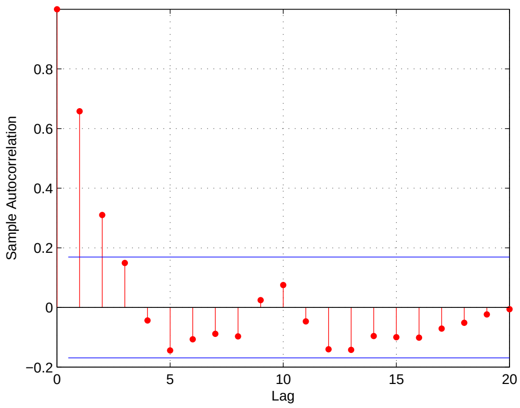
Figure 6: Trajectories and autocorrelation functions of HF and LF factors.



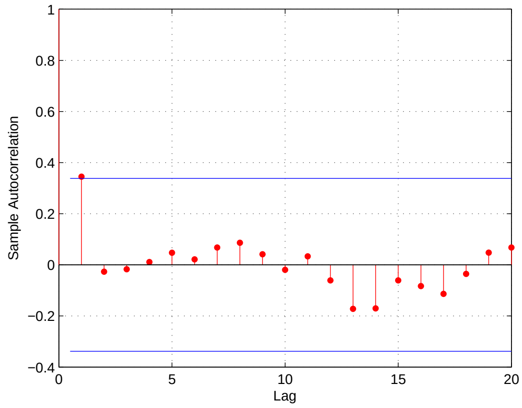
(a) HF factor: estimated values.



(b) LF factor: estimated values.



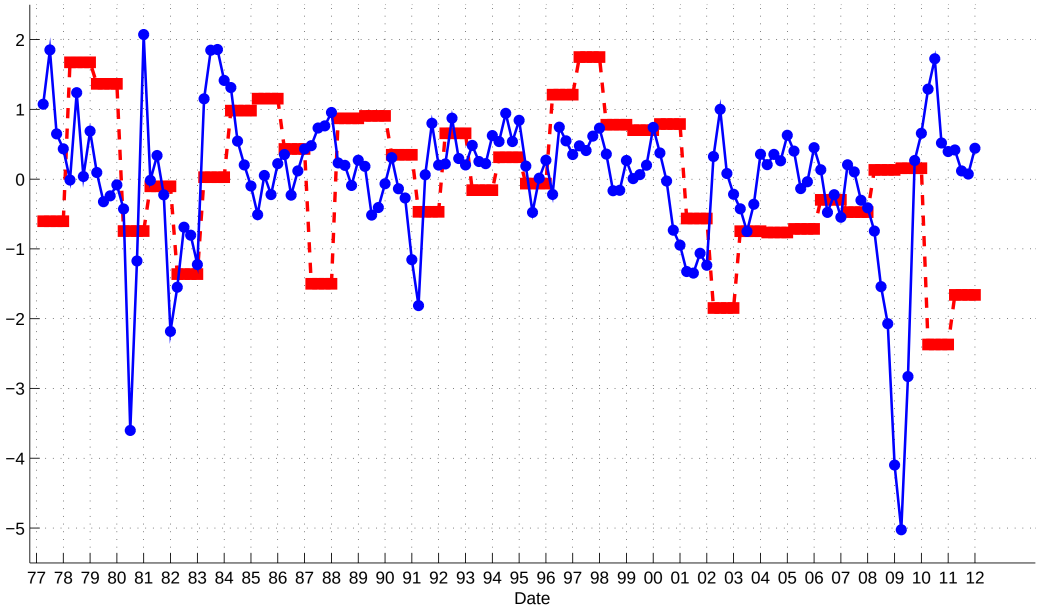
(c) HF factor: autocorrelation function.



(d) LF factor: autocorrelation function.

Panel (a) displays the time series of estimated values of the HF factor. Panel (b) displays the time series of estimated values of the LF factor. Panel (c) displays the empirical autocorrelation function of the estimated values of the HF factor. Panel (d) displays the empirical autocorrelation function of the estimated values of the LF factor. The horizontal lines are asymptotic 95% confidence bands. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of the factor model is 1977.Q1-2011.Q4.

Figure 7: Trajectories of HF and LF factors.



The Figure displays the time series of estimated values of the HF factor (blue circles) an LF factors (red squares). For each year we represent the LF factor as 4 squares corresponding to the 4 quarters, assuming the same value. The factors are estimated from the panel of 42 GDP sectors and 117 industrial production indexes considered by Foerster, Sarte, and Watson (2011), using a mixed frequency factor model with $K_H = K_L = 1$. The sample period for the estimation of the factor model is 1977.Q1-2011.Q4.

APPENDIX A: Restrictions on the factor dynamics

In this Appendix we derive restrictions on the structural VAR parameters of the factor dynamics implied by *i)* the factor normalization and *ii)* stationarity.

A.1 Implied restrictions from factor normalization

The unconditional variance-covariance matrix of the vector of stacked factors $(f_{1,t}' \;\; f_{2,t}' \;\; g_t')'$ is (see equation (2)):

$$V = V \begin{pmatrix} f_{1,t} \\ f_{2,t} \\ g_t \end{pmatrix} = \begin{bmatrix} I_{K_H} & \Phi & 0 \\ \Phi' & I_{K_H} & 0 \\ 0 & 0 & I_{K_L} \end{bmatrix},$$

where Φ is the covariance between $f_{1,t}$ and $f_{2,t}$. Moreover, the factor dynamics is given by the structural VAR(1) model (see equation (3)):

$$\Gamma \begin{bmatrix} f_{1,t} \\ f_{2,t} \\ g_t \end{bmatrix} = R \begin{bmatrix} f_{1,t-1} \\ f_{2,t-1} \\ g_{t-1} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \\ w_t \end{bmatrix}, \tag{A.1}$$

where:

$$\Gamma = \begin{bmatrix} I_{K_H} & 0 & 0 \\ -R_H & I_{K_H} & 0 \\ 0 & 0 & I_{K_L} \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & R_H & A_1 \\ 0 & 0 & A_2 \\ M_1 & M_2 & R_L \end{bmatrix},$$

and $(v_{1,t}', v_{2,t}', w_t')'$ is a multivariate white noise process with mean 0 and variance-covariance matrix (see equation (4)):

$$\Sigma = V \begin{pmatrix} v_{1,t} \\ v_{2,t} \\ w_t \end{pmatrix} = \begin{bmatrix} \Sigma_H & 0 & \Sigma_{HL,1} \\ & \Sigma_H & \Sigma_{HL,2} \\ & & \Sigma_L \end{bmatrix}.$$

By computing the variance on both sides of equation (A.1) we get:

$$\Gamma V \Gamma' = RVR' + \Sigma. \tag{A.2}$$

By matrix multiplication:

$$\Gamma V \Gamma' = \begin{bmatrix} I_{K_H} & \Phi - R_H' & 0 \\ \Phi' - R_H & I_{K_H} - R_H\Phi - \Phi' R_H' + R_H R_H' & 0 \\ 0 & 0 & I_{K_L} \end{bmatrix},$$

and:

$$RVR' = \begin{bmatrix} R_H R_H' + A_1 A_1' & A_1 A_2' & R_H(\Phi' M_1' + M_2') + A_1 R_L' \\ A_2 A_1' & A_2 A_2' & A_2 R_L' \\ (M_1\Phi + M_2) R_H' + R_L A_1' & R_L A_2' & M_1 M_1' + M_2 M_2' + M_1 \Phi M_2' + M_2 \Phi' M_1' + R_L R_L' \end{bmatrix}.$$

Hence from equation (A.2) we get the following system of equations:

$$I_{K_H} = R_H R_H' + A_1 A_1' + \Sigma_H, \tag{A.3}$$

$$I_{K_H} - R_H \Phi - \Phi' R_H' + R_H R_H' = A_2 A_2' + \Sigma_H, \tag{A.4}$$

$$I_{K_L} = M_1 M_1' + M_2 M_2' + M_1 \Phi M_2' + M_2 \Phi' M_1' + R_L R_L' + \Sigma_L, \tag{A.5}$$

$$\Phi - R_H' = A_1 A_2', \tag{A.6}$$

$$0 = R_H(\Phi' M_1' + M_2') + A_1 R_L' + \Sigma_{HL,1}, \tag{A.7}$$

$$0 = A_2 R_L' + \Sigma_{HL,2}. \tag{A.8}$$

These equations imply:

$$\Sigma_H = I_{K_H} - R_H R_H' - A_1 A_1', \tag{A.9}$$

$$\Sigma_H = I_{K_H} - R_H \Phi - \Phi' R_H' + R_H R_H' - A_2 A_2', \tag{A.10}$$

$$\Sigma_L = I_{K_L} - M_1 M_1' - M_1 \Phi M_2' - M_2 \Phi' M_1' - M_2 M_2' - R_L R_L', \tag{A.11}$$

$$\Phi = R_H' + A_1 A_2', \tag{A.12}$$

$$\Sigma_{HL,1} = -R_H(\Phi' M_1' + M_2') - A_1 R_L', \tag{A.13}$$

$$\Sigma_{HL,2} = -A_2 R_L'. \tag{A.14}$$

Let θ be the vector containing the elements of matrices R_H, R_L, A_1, A_2, M_1 and M_2 in the structural VAR(1) model. Equation (A.12) expresses the stationary autocovariance matrix Φ of the HF factor as function of θ (more precisely, of R_H and A_1, A_2). Equations (A.9), (A.11), (A.13) and (A.14) express the variance-covariance matrix Σ of the factor innovations as a function of θ. Finally, equation (A.10), together with equations (A.9) and (A.12), implies a restriction on vector θ:

$$A_1 A_1' - R_H A_1 A_2' - A_1 A_2' R_H' - A_2 A_2' = 0. \tag{A.15}$$

Thus, the factor dynamics is characterized by parameter matrices R_H, R_L, A_1, A_2, M_1 and M_2, which are subject to restriction (A.15),

Let us now discuss restriction (A.15) in the case of single HF and LF factors, i.e. $K_H = K_L = 1$. Equation (A.15) becomes:

$$A_1^2 - 2R_H A_1 A_2 - A_2^2 = 0, \tag{A.16}$$

where A_1, A_2 and R_H are scalars. This equation yields two solutions for A_1 as a function of A_2 and R_H:

$$A_1 \quad = \quad A_2 \left(R_H \pm \sqrt{1 + R_H^2} \right).$$

A.2 Stationarity conditions

The stationarity conditions are deduced from the reduced form of the VAR(1) dynamics in (A.1), that is:

$$\begin{bmatrix} f_{1,t} \\ f_{2,t} \\ g_t \end{bmatrix} = \Gamma^{-1}R \begin{bmatrix} f_{1,t-1} \\ f_{2,t-1} \\ g_{t-1} \end{bmatrix} + \zeta_t$$

where:

$$\Gamma^{-1}R = \begin{bmatrix} 0 & R_H & A_1 \\ 0 & R_H^2 & R_H A_1 + A_2 \\ M_1 & M_2 & R_L \end{bmatrix},$$

and $\zeta_t = \Gamma^{-1}(v'_{1,t}, v'_{2,t}, w'_t)'$ is a zero-mean white noise process with variance-covariance matrix

$$\Sigma_\zeta = \Gamma^{-1}\Sigma(\Gamma^{-1})' = \begin{bmatrix} \Sigma_H & \Sigma_H R'_H & \Sigma_{HL,1} \\ R_H\Sigma_H & R_H\Sigma_H R'_H + \Sigma_H & \Sigma_{HL,2} + R_H\Sigma_{HL,1} \\ \Sigma'_{HL,1} & \Sigma'_{HL,2} + \Sigma'_{HL,1}R'_H & \Sigma_L \end{bmatrix}. \tag{A.17}$$

The stationarity condition is: the eigenvalues of matrix $\Gamma^{-1}R$ are smaller than 1 in modulus. When either $M_1 = M_2 = 0$, or $A_1 = A_2 = 0$, the stationarity condition becomes: the eigenvalues of matrices R_H and R_L are smaller than 1 in modulus.

APPENDIX B: Identification

B.1 Proof of Proposition 1

By replacing equation (6) into model (1), we get

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ y_t \end{bmatrix} = \begin{bmatrix} \Lambda A_{11} + \Delta_1 A_{31} & \Lambda A_{12} + \Delta_1 A_{32} & \Lambda A_{13} + \Delta_1 A_{33} \\ \Lambda A_{21} + \Delta_2 A_{31} & \Lambda A_{22} + \Delta_2 A_{32} & \Lambda A_{23} + \Delta_2 A_{33} \\ \Omega_1 A_{11} + \Omega_2 A_{21} + B A_{31} & \Omega_1 A_{12} + \Omega_2 A_{22} + B A_{32} & \Omega_1 A_{13} + \Omega_2 A_{23} + B A_{33} \end{bmatrix} \begin{bmatrix} \tilde{f}_{1,t} \\ \tilde{f}_{2,t} \\ \tilde{g}_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ u_t \end{bmatrix}. \tag{B.1}$$

This factor model satisfies the restrictions in the loading matrix displayed in equation (1) if, and only if,

$$\Lambda A_{12} + \Delta_1 A_{32} = 0, \tag{B.2}$$

$$\Lambda A_{21} + \Delta_2 A_{31} = 0, \tag{B.3}$$

$$\Lambda A_{11} + \Delta_1 A_{31} = \Lambda A_{22} + \Delta_2 A_{32}. \tag{B.4}$$

Let us assume that $\begin{bmatrix} \Lambda \vdots \Delta_1 \end{bmatrix}$ is full column rank for N_H sufficiently large (the argument for the case in which $\begin{bmatrix} \Lambda \vdots \Delta_2 \end{bmatrix}$ is full column rank is similar). Equation (B.2) can be written as a linear homogeneous system of equations for the elements of matrices A_{12} and A_{32}:

$$\begin{bmatrix} \Lambda \vdots \Delta_1 \end{bmatrix} \begin{bmatrix} A_{12} \\ A_{32} \end{bmatrix} = 0.$$

Since $\begin{bmatrix} \Lambda \vdots \Delta_1 \end{bmatrix}$ is full column rank, it follows that

$$A_{12} = 0 \text{ and } A_{32} = 0. \tag{B.5}$$

Then, equation (B.4) becomes $\Lambda(A_{11} - A_{22}) + \Delta_1 A_{31} = 0$, that is:

$$\left[\Lambda \vdots \Delta_1\right] \begin{bmatrix} A_{11} - A_{22} \\ A_{31} \end{bmatrix} = 0. \tag{B.6}$$

Since $\left[\Lambda \vdots \Delta_1\right]$ is full column rank it follows that:

$$A_{11} = A_{22}, \tag{B.7}$$

$$A_{31} = 0. \tag{B.8}$$

Replacing the last equation in (B.3), and using that matrix Λ is full rank, we get:

$$A_{21} = 0. \tag{B.9}$$

Thus, the transformation of the factors that is compatible with the restrictions on the loading matrix in equation (1) is:

$$\begin{bmatrix} f_{1,t} \\ f_{2,t} \\ g_t \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} \tilde{f}_{1,t} \\ \tilde{f}_{2,t} \\ \tilde{g}_t \end{bmatrix}, \qquad A_{11} = A_{22}.$$

We can invert this transformation and write:

$$\begin{aligned} \tilde{f}_{1,t} &= A_{11}^{-1} f_{1,t} - A_{11}^{-1} A_{13} A_{33}^{-1} g_t, \\ \tilde{f}_{2,t} &= A_{22}^{-1} f_{2,t} - A_{22}^{-1} A_{23} A_{33}^{-1} g_t, \\ \tilde{g}_t &= A_{33}^{-1} g_t. \end{aligned}$$

The transformed factors satisfy the normalization restrictions in (2) if, and only if,

$$\begin{aligned} Cov(\tilde{f}_{1,t},\ \tilde{g}_t) &= -A_{11}^{-1}A_{13}A_{33}^{-1}(A_{33}^{-1})' = 0, && \text{(B.10)} \\ Cov(\tilde{f}_{2,t},\ \tilde{g}_t) &= -A_{22}^{-1}A_{23}A_{33}^{-1}(A_{33}^{-1})' = 0, && \text{(B.11)} \\ V(\tilde{f}_{1,t}) &= A_{11}^{-1}(A_{11}^{-1})' + A_{11}^{-1}A_{13}A_{33}^{-1}(A_{33}^{-1})'A_{13}'(A_{11}^{-1})' = I_{K_H}, && \text{(B.12)} \\ V(\tilde{f}_{2,t}) &= A_{22}^{-1}(A_{22}^{-1})' + A_{22}^{-1}A_{23}A_{33}^{-1}(A_{33}^{-1})'A_{23}'(A_{22}^{-1})' = I_{K_H}, && \text{(B.13)} \\ V(\tilde{g}_t) &= A_{33}^{-1}(A_{33}^{-1})' = I_{K_L}. && \text{(B.14)} \end{aligned}$$

Since the matrices $A_{11} = A_{22}$ and A_{33} are nonsingular, equations (B.10) and (B.11) imply

$$A_{13} = A_{23} = 0. \tag{B.15}$$

Then from equations (B.12) - (B.15), we get that matrices $A_{11} = A_{22}$ and A_{33} are orthogonal.

Q.E.D.

B.2 Proof of Proposition 2

If $\Delta_1 = \Delta_2 = 0$ in the DGP, from (B.1) we get:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ y_t \end{bmatrix} = \begin{bmatrix} \Lambda A_{11} & \Lambda A_{12} & \Lambda A_{13} \\ \Lambda A_{21} & \Lambda A_{22} & \Lambda A_{23} \\ \Omega_1 A_{11} + \Omega_2 A_{21} + BA_{31} & \Omega_1 A_{12} + \Omega_2 A_{22} + BA_{32} & \Omega_1 A_{13} + \Omega_2 A_{23} + BA_{33} \end{bmatrix} \begin{bmatrix} \tilde{f}_{1,t} \\ \tilde{f}_{2,t} \\ \tilde{g}_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ u_t \end{bmatrix}. \tag{B.16}$$

The restrictions on the loading matrices imply:

$$\Lambda A_{12} = 0, \qquad \Lambda A_{21} = 0, \qquad \Lambda A_{11} = \Lambda A_{22}.$$

Since Λ is full column rank, it follows $A_{12} = 0$, $A_{21} = 0$ and $A_{11} = A_{22}$. In the transformed model (B.16), the loadings of the LF factor on the HF data are:

$$\tilde{\Delta}_1 = \Lambda A_{13}, \qquad \tilde{\Delta}_2 = \Lambda A_{23}, \tag{B.17}$$

that are spanned by Λ. By Assumption 1, it follows $\tilde{\Delta}_1 = 0$ and $\tilde{\Delta}_2 = 0$, and hence $A_{13} = 0$ and $A_{23} = 0$. Then from (6):

$$\begin{cases} \tilde{f}_{1,t} &= A_{11}^{-1} f_{1,t} \\ \tilde{f}_{2,t} &= A_{22}^{-1} f_{2,t} \\ \tilde{g}_t &= A_{33}^{-1}(g_t - A_{31}A_{11}^{-1} f_{1,t} - A_{32}A_{22}^{-1} f_{2,t}) \end{cases} . \tag{B.18}$$

Then:

$$\begin{aligned} 0 &= Cov(\tilde{g}_t, \tilde{f}_{1,t}) = -A_{33}^{-1}(A_{31}A_{11}^{-1} + A_{32}A_{22}^{-1}\Phi')(A_{11}^{-1})', \\ 0 &= Cov(\tilde{g}_t, \tilde{f}_{2,t}) = -A_{33}^{-1}(A_{31}A_{11}^{-1}\Phi + A_{32}A_{22}^{-1})(A_{22}^{-1})'. \end{aligned}$$

Thus, we get:

$$\begin{aligned} A_{31}A_{11}^{-1} + A_{32}A_{22}^{-1}\Phi' &= 0, \\ A_{31}A_{11}^{-1}\Phi + A_{32}A_{22}^{-1} &= 0, \end{aligned} \tag{B.19}$$

which implies:

$$A_{32}A_{22}^{-1}[I_{K_H} - \Phi'\Phi] = 0.$$

Since the variance-covariance matrix of the factors in (2) is positive definite, the matrix $I_{K_H} - \Phi'\Phi$ is invertible. Then, we get $A_{32} = 0$. From (B.19) it follows $A_{31} = 0$.

Q.E.D.

APPENDIX C: Large sample properties

C.1 Proof of Proposition 3

Let us introduce a new notation for the matrices of HF and LF observations, factors, and errors, respectively:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \qquad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \qquad \hat{F} = \begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{bmatrix}, \qquad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix},$$

and the residuals matrices:

$$\Xi = \begin{bmatrix} \Xi_1 \\ \Xi_2 \end{bmatrix} = \begin{bmatrix} M_G X_1 \\ M_G X_2 \end{bmatrix}, \qquad \tilde{\Xi} = \begin{bmatrix} \tilde{\Xi}_1 \\ \tilde{\Xi}_2 \end{bmatrix} = \begin{bmatrix} M_{\tilde{G}} X_1 \\ M_{\tilde{G}} X_2 \end{bmatrix},$$

where $M_G = I - P_G$, with $P_G = G(G'G)^{-1}G'$, and $M_{\tilde{G}} = I - P_{\tilde{G}}$, with $P_{\tilde{G}} = \tilde{G}(\tilde{G}'\tilde{G})^{-1}\tilde{G}'$. We define:

$$\begin{aligned}
&F^* = [F_1 \;\; F_2], \qquad \hat{F}^* = [\hat{F}_1 \;\; \hat{F}_2], \\
&\Delta = [\Delta_1 \;\; \Delta_2], \qquad \Omega = [\Omega_1 \;\; \Omega_2], \\
&G^* = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix} = I_2 \otimes G, \qquad \hat{G}^* = \begin{bmatrix} \hat{G} & 0 \\ 0 & \hat{G} \end{bmatrix} = I_2 \otimes \hat{G}, \\
&P_{G^*} = \begin{bmatrix} P_G & 0 \\ 0 & P_G \end{bmatrix}, \qquad M_{G^*} = \begin{bmatrix} M_G & 0 \\ 0 & M_G \end{bmatrix}.
\end{aligned}$$

The hat and tilde refer to the estimates in the current and previous iterations, respectively, in the iterative estimation procedure.

The model (1) can be written as:

$$\begin{aligned}
X_1 &= F_1\Lambda' + G\Delta_1' + \varepsilon_1, \\
X_2 &= F_2\Lambda' + G\Delta_2' + \varepsilon_2, \\
Y &= F_1\Omega_1' + F_2\Omega_2' + GB' + u,
\end{aligned}$$

and, more compactly, as:

$$X = F\Lambda' + G^*\Delta' + \varepsilon, \tag{C.1}$$

$$Y = F^*\Omega' + GB' + u. \tag{C.2}$$

C.1.1 The exact recursive equation in step 1

The first step of the iterative estimation procedure consists in the estimation by PCA of the HF factor from the HF data, given the estimated LF factor from the previous iteration. By reordering of the data, from equation (11) we have:

$$\frac{1}{2N_H T}\tilde{\Xi}\tilde{\Xi}'\hat{F} = \hat{F}\hat{V}_F. \tag{C.3}$$

The matrix $\tilde{\Xi}$ can be decomposed as:

$$\begin{aligned}\tilde{\Xi} &= M_{\tilde{G}^*}X = M_{G^*}X + (M_{\tilde{G}^*} - M_{G^*})X \\ &= M_{G^*}(F\Lambda' + \varepsilon) - (P_{\tilde{G}^*} - P_{G^*})X \\ &= F\Lambda' + e - (P_{\tilde{G}^*} - P_{G^*})X,\end{aligned}$$

where:

$$e = \varepsilon - P_{G^*}(F\Lambda' + \varepsilon). \tag{C.4}$$

Therefore matrix $\tilde{\Xi}\tilde{\Xi}'$ can be expressed as:

$$\begin{aligned}\tilde{\Xi}\tilde{\Xi}' = {} & F\Lambda'\Lambda F' + \Big\{ee' + (P_{\tilde{G}^*} - P_{G^*})XX'(P_{\tilde{G}^*} - P_{G^*}) \\ & + F\Lambda'e' + e\Lambda F' \\ & - F\Lambda'X'(P_{\tilde{G}^*} - P_{G^*}) - (P_{\tilde{G}^*} - P_{G^*})X\Lambda F' \\ & - eX'(P_{\tilde{G}^*} - P_{G^*}) - (P_{\tilde{G}^*} - P_{G^*})Xe'\Big\}.\end{aligned} \tag{C.5}$$

The equation (C.3) can be written as:

$$\hat{F}\hat{V}_F - F\left(\frac{\Lambda'\Lambda}{N_H}\right)\left(\frac{F'\hat{F}}{2T}\right) = \frac{1}{2N_H T}\left\{...\right\}\hat{F}, \tag{C.6}$$

where the terms in the curly brackets are the same as in equation (C.5). Since $\frac{F'\hat{F}}{2T}$ is invertible w.p.a. 1 from Lemma S.1, then:

$$\hat{F}\hat{V}_F\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} - F = \frac{1}{2N_H T}\left\{...\right\}\hat{F}\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}.$$

Since $\hat{V}_F$ is invertible w.p.a. 1 from Lemma S.2 we can define:

$$\hat{H}_F = \left(\frac{\Lambda'\Lambda}{N_H}\right)\left(\frac{F'\hat{F}}{2T}\right)\hat{V}_F^{-1}.$$

Then $\hat{H}_F$ is invertible w.p.a. 1 and:

$$\hat{H}_F^{-1} = \hat{V}_F\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}.$$

We get:

$$\begin{aligned}\hat{F}\hat{H}_F^{-1} - F = \frac{1}{2N_H T}\Big\{ & ee'\hat{F} + F\Lambda'e'\hat{F} + e\Lambda F'\hat{F} \\ & -eX'(P_{\tilde{G}^*} - P_{G^*})\hat{F} - (P_{\tilde{G}^*} - P_{G^*})Xe'\hat{F} \\ & -F\Lambda'X'(P_{\tilde{G}^*} - P_{G^*})\hat{F} - (P_{\tilde{G}^*} - P_{G^*})X\Lambda F'\hat{F} \\ & +(P_{\tilde{G}^*} - P_{G^*})XX'(P_{\tilde{G}^*} - P_{G^*})\hat{F}\Big\}\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}.\end{aligned} \tag{C.7}$$

C.1.2 The exact recursive equation in step 2

The second step of the iterative estimation procedure consists in the estimation by PCA of the LF factor from the LF data, given the estimated HF factor from the first step (see equation (12)):

$$\frac{1}{N_L T}\hat{\Psi}\hat{\Psi}'\hat{G} = \hat{G}\hat{V}_G. \tag{C.8}$$

The matrix $\hat{\Psi}$ can be decomposed as:

$$\begin{aligned}\hat{\Psi} &= M_{\hat{F}^*}Y = (I - P_{F^*})Y - (P_{\hat{F}^*} - P_{F^*})Y \\ &= GB' + v - (P_{\hat{F}^*} - P_{F^*})Y,\end{aligned}$$

where:

$$v = u - P_{F^*}(GB' + u). \tag{C.9}$$

Therefore matrix $\hat{\Psi}\hat{\Psi}'$ can be expressed as:

$$\begin{aligned}\hat{\Psi}\hat{\Psi}' &= GB'BG' + \Big\{ vv' + (P_{\hat{F}^*} - P_{F^*})YY'(P_{\hat{F}^*} - P_{F^*}) \\ &\quad + GB'v' + vBG' \\ &\quad - GB'Y'(P_{\hat{F}^*} - P_{F^*}) - (P_{\hat{F}^*} - P_{F^*})YBG' \\ &\quad - vY'(P_{\hat{F}^*} - P_{F^*}) - (P_{\hat{F}^*} - P_{F^*})Yv' \Big\}.\end{aligned}$$

The equation (C.8) can be written as:

$$\hat{G}\hat{V}_G - G\left(\frac{B'B}{N_L}\right)\left(\frac{G'\hat{G}}{T}\right) = \frac{1}{N_L T}\Big\{ ... \Big\}\hat{G}. \tag{C.10}$$

Since $\dfrac{G'\hat{G}}{T}$ is invertible from Lemma S.1, then:

$$\hat{G}\hat{V}_G\left(\frac{G'\hat{G}}{T}\right)^{-1}\left(\frac{B'B}{N_L}\right)^{-1} - G = \frac{1}{N_L T}\Big\{ ... \Big\}\hat{G}\left(\frac{G'\hat{G}}{T}\right)^{-1}\left(\frac{B'B}{N_L}\right)^{-1}.$$

Since $\hat{V}_G$ is invertible w.p.a. 1 from Lemma S.2 we can define:

$$\hat{H}_G = \left(\frac{B'B}{N_L}\right)\left(\frac{G'\hat{G}}{T}\right)\hat{V}_G^{-1}.$$

Then $\hat{H}_G$ is invertible w.p.a. 1 and:

$$\hat{H}_G^{-1} = \hat{V}_G \left(\frac{G'\hat{G}}{T} \right)^{-1} \left(\frac{B'B}{N_L} \right)^{-1}.$$

We get:

$$\begin{aligned} \hat{G}\hat{H}_G^{-1} - G &= \frac{1}{N_L T} \Big\{ vv'\hat{G} + GB'v'\hat{G} + vBG'\hat{G} \\ &\quad - vY'(P_{\hat{F}^*} - P_{F^*})\hat{G} - (P_{\hat{F}^*} - P_{F^*})Yv'\hat{G} \\ &\quad - GB'Y'(P_{\hat{F}^*} - P_{F^*})\hat{G} - (P_{\hat{F}^*} - P_{F^*})YBG'\hat{G} \\ &\quad + (P_{\hat{F}^*} - P_{F^*})YY'(P_{\hat{F}^*} - P_{F^*})\hat{G} \Big\} \left(\frac{G'\hat{G}}{T} \right)^{-1} \left(\frac{B'B}{N_L} \right)^{-1}. \qquad \text{(C.11)} \end{aligned}$$

Equations (C.7) and (C.11) are a system of nonlinear implicit equations, which define the new estimates $\hat{F}$ and $\hat{G}$ in terms of the old estimate $\tilde{G}$. In the next two subsections we linearize these equations around the true factor values F and G.

C.1.3 The linearized equation in step 1

Let us define matrix $\tilde{H}_{G^*}$ as:

$$\tilde{H}_{G^*} = \begin{bmatrix} \tilde{H}_G & 0 \\ 0 & \tilde{H}_G \end{bmatrix},$$

where $\tilde{H}_G = \left(\frac{B'B}{N_L} \right) \left(\frac{G'\tilde{G}}{T} \right) \tilde{V}_G^{-1}$ and $\tilde{V}_G$ is the matrix of eigenvalues in the PCA problem defining $\tilde{G}$.

Lemma C.1. *We have:*

$$\begin{aligned} \hat{F}\hat{H}_F^{-1} - F &= \eta_F - M_{G^*}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)D' - G^*(G^{*\prime}G^*)^{-1}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'F \\ &\quad - F\left(\frac{\Lambda'\Lambda}{N_H} \right) D \left[\frac{1}{2T}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'F \right] \left(\frac{F'F}{2T} \right)^{-1} \left(\frac{\Lambda'\Lambda}{N_H} \right)^{-1} \\ &\quad + \mathscr{R}_F(\hat{F}, \tilde{G}), \qquad \text{(C.12)} \end{aligned}$$

where

$$D = \lim_{N_H\to\infty}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\left(\frac{\Lambda'\Delta}{N_H}\right) = [D_1 \;\; D_2],$$

the term

$$\eta_F = \frac{1}{2N_HT}\left(ee'\hat{F} + F\Lambda'e'\hat{F}\right)\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} + \frac{1}{N_H}e\Lambda\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} \tag{C.13}$$

is such that

$$\|\eta_F\|/\sqrt{2T} = O_p\left(\frac{1}{\sqrt{\min(N_H, 2T)}}\right), \tag{C.14}$$

and the reminder term $\mathscr{R}_F(\hat{F}, \tilde{G})$ is such that

$$\begin{aligned}\|\mathscr{R}_F(\hat{F}, \tilde{G})\|/\sqrt{2T} &= O_p\bigg(\frac{1}{\sqrt{\min(N_H, T)}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|/\sqrt{2T} + (\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|/\sqrt{2T})^2\bigg) \\ &\quad + O_p\bigg(\frac{1}{2T}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|\|\hat{F}\hat{H}_F^{-1} - F\|\bigg)\end{aligned} \tag{C.15}$$

To prove Lemma C.1, we need the following two lemmas, which are proved in the supplementary material:

Lemma C.2. *We have:*

(a) $\frac{1}{N_HT}\|\varepsilon\varepsilon'\| = O_p\left(\frac{1}{\sqrt{\min(N_H, T)}}\right)$.

(b) $\frac{1}{N_HT}\|F\Lambda'\varepsilon'\| = O_p\left(\frac{1}{\sqrt{N_H}}\right)$.

(c) $\frac{1}{N_H}\|\varepsilon\Lambda\| = O_p\left(\sqrt{\frac{T}{N_H}}\right)$.

(d) $\frac{1}{N_HT}\|\varepsilon\Delta G^{*\prime}\| = O_p\left(\frac{1}{\sqrt{N_H}}\right)$.

Lemma C.3. *We have:*

(a) $\frac{1}{N_HT}\|ee'\| = O_p\left(\frac{1}{\sqrt{\min(N_H, T)}}\right)$.

(b) $\frac{1}{N_H T}\|F\Lambda' e'\| = O_p\left(\frac{1}{\sqrt{\min(N_H,T)}}\right)$.

(c) $\frac{1}{N_H}\|e\Lambda\| = O_p\left(\sqrt{\frac{T}{N_H}}\right)$.

(d) $\frac{1}{N_H T}\|e\Delta G^{*\prime}\| = O_p\left(\frac{1}{\sqrt{\min(N_H,T)}}\right)$.

(e) $\frac{1}{N_H T}\|e\varepsilon'\| = O_p\left(\frac{1}{\sqrt{\min(N_H,T)}}\right)$.

Proof of Lemma C.1: *i)* Let us first show the decomposition in equation (C.12). By rearranging the terms in the RHS of equation (C.7) we get:

$$\begin{aligned}\hat{F}\hat{H}_F^{-1} - F &= \eta_F - \frac{1}{2T}\left[F\left(\frac{X\Lambda}{N_H}\right)'(P_{\tilde{G}^*} - P_{G^*})\hat{F}\right]\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} \\ &\quad -(P_{\tilde{G}^*} - P_{G^*})\left(\frac{X\Lambda}{N_H}\right)\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} + \mathscr{R}_{1,F}(\hat{F},\tilde{G}), \qquad \text{(C.16)}\end{aligned}$$

where η_F is defined in (C.13), and

$$\begin{aligned}\mathscr{R}_{1,F}(\hat{F},\tilde{G}) &= -\frac{1}{2N_H T}\Big\{eX'(P_{\tilde{G}^*} - P_{G^*}) - (P_{\tilde{G}^*} - P_{G^*})Xe' \\ &\quad +(P_{\tilde{G}^*} - P_{G^*})XX'(P_{\tilde{G}^*} - P_{G^*})\Big\}\hat{F}\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}. \qquad \text{(C.17)}\end{aligned}$$

Let us now consider the matrix $X\Lambda/N_H$ in the RHS of equation (C.16). By using the model of X in equation (C.1):

$$\frac{X\Lambda}{N_H} = F\left(\frac{\Lambda'\Lambda}{N_H}\right) + G^*\left(\frac{\Delta'\Lambda}{N_H}\right) + \left(\frac{\varepsilon\Lambda}{N_H}\right). \qquad \text{(C.18)}$$

Thus:

$$\begin{aligned}\left(\frac{X\Lambda}{N_H}\right)\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} &= F + G^*\left(\frac{\Delta'\Lambda}{N_H}\right)\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} + \left(\frac{\varepsilon\Lambda}{N_H}\right)\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} \\ &= Z + G^*\left[\left(\frac{\Delta'\Lambda}{N_H}\right)\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} - D'\right] + \left(\frac{\varepsilon\Lambda}{N_H}\right)\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}, \qquad \text{(C.19)}\end{aligned}$$

where:

$$Z \;=\; F + G^{*}D'. \tag{C.20}$$

Then the second and third terms in the RHS of equation (C.16) become:

$$\begin{aligned}&\frac{1}{2T}\left[F\left(\frac{X\Lambda}{N_H}\right)'(P_{\tilde{G}^*}-P_{G^*})\hat{F}\right]\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}+(P_{\tilde{G}^*}-P_{G^*})\left(\frac{X\Lambda}{N_H}\right)\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\\&=\frac{1}{2T}\left[F\left(\frac{\Lambda'\Lambda}{N_H}\right)Z'(P_{\tilde{G}^*}-P_{G^*})\hat{F}\right]\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}+(P_{\tilde{G}^*}-P_{G^*})Z+\mathscr{R}_{2,F}(\hat{F},\tilde{G}),\end{aligned} \tag{C.21}$$

where:

$$\begin{aligned}\mathscr{R}_{2,F}(\hat{F},\tilde{G}) \;=\;& \frac{1}{2T}F\left(\frac{\Lambda'\Lambda}{N_H}\right)\left[\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\left(\frac{\Lambda'\Delta}{N_H}\right)-D\right]G^{*\prime}(P_{\tilde{G}^*}-P_{G^*})\hat{F}\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\\&+\frac{1}{2T}F\left(\frac{\Lambda'\varepsilon'}{N_H}\right)(P_{\tilde{G}^*}-P_{G^*})\hat{F}\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\\&+(P_{\tilde{G}^*}-P_{G^*})\left\{G^*\left[\left(\frac{\Delta'\Lambda}{N_H}\right)\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}-D'\right]+\left(\frac{\varepsilon\Lambda}{N_H}\right)\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\right\}.\end{aligned} \tag{C.22}$$

Let us now consider the first two terms in the RHS of equation (C.21). In order to linearize the term $P_{\tilde{G}^*} - P_{G^*}$, we need the following Lemma C.4. We use the *operator norm* $\|\cdot\|_{op}$, which, for the generic $(m \times n)$ matrix A, is defined as (see, e.g., Horn and Johnson (2013)):

$$\|A\|_{op} = \sup_{\|x\|=1} \|Ax\| \,. \tag{C.23}$$

Lemma C.4. *Let $\hat{A}$ and A be two $m \times n$ matrices, where A is full column rank and $\hat{A}$. Then, if*

$$\|(A'A)^{-1}\|_{op}^{1/2}\, \|\hat{A}-A\|_{op} \;<\; \sqrt{1+\varrho}-1,$$

for some $\varrho \in (0,1)$, we have:

$$P_{\hat{A}} - P_A \;=\; M_A(\hat{A}-A)(A'A)^{-1}A' + A(A'A)^{-1}(\hat{A}-A)'M_A + \mathscr{R}_P(\hat{A},A),$$

where $P_A = A(A'A)^{-1}A'$ and $M_A = I_m - P_A$, and the reminder term $\mathscr{R}_P(\hat{A}, A)$ is such that:

$$\|\mathscr{R}_P(\hat{A}, A)\|_{op} \leq C\left(\|(A'A)^{-1}\|_{op} + \|(A'A)^{-1}\|_{op}^2\right)\|\hat{A} - A\|_{op}^2,$$

with constant $C < \infty$ is independent of A and $\hat{A}$, but may depend on ϱ.

The proof of Lemma C.4 is given in the supplementary material. By using $P_{\tilde{G}^* \tilde{H}_{G^*}^{-1}} = P_{\tilde{G}^*}$ and applying Lemma C.4 with $\hat{A} = \tilde{G}^*\tilde{H}_{G^*}^{-1}$ and $A = G^*$, we have:

$$\begin{aligned} P_{\tilde{G}^*} - P_{G^*} &= M_{G^*}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)(G^{*\prime}G^*)^{-1}G^{*\prime} \\ &\quad + G^*(G^{*\prime}G^*)^{-1}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'M_{G^*} + \mathscr{R}_P(\tilde{G}^*, G^*), \end{aligned} \tag{C.24}$$

where

$$\|\mathscr{R}_P(\tilde{G}^*, G^*)\|_{op} = O(\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|_{op}^2\|(G^{*\prime}G^*)^{-1}G^{*\prime}\|_{op}^2). \tag{C.25}$$

Then:

$$\begin{aligned} (P_{\tilde{G}^*} - P_{G^*})Z &= M_{G^*}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)D' \\ &\quad + G^*(G^{*\prime}G^*)^{-1}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'F + \mathscr{R}_{3,F}(\tilde{G}), \end{aligned} \tag{C.26}$$

where:

$$\begin{aligned} \mathscr{R}_{3,F}(\tilde{G}) &= M_{G^*}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)(G^{*\prime}G^*)^{-1}G^{*\prime}F \\ &\quad - G^*(G^{*\prime}G^*)^{-1}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'P_{G^*}F + \mathscr{R}_P(\tilde{G}^*, G^*)Z, \end{aligned} \tag{C.27}$$

and:

$$\begin{aligned} Z'(P_{\tilde{G}^*} - P_{G^*})\hat{F} &= D(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'F\hat{H}_F \\ &\quad + \left\{F'(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)(G^{*\prime}G^*)^{-1}G^{*\prime}\hat{F} + \mathscr{R}_{3,F}(\tilde{G})'\hat{F}\right\} \\ &\quad + D(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'(\hat{F} - F\hat{H}_F - P_{G^*}\hat{F}). \end{aligned} \tag{C.28}$$

Rewriting $\hat{F} - F\hat{H}_F - P_{G^*}\hat{F}$ as:

$$\begin{aligned}\hat{F} - F\hat{H}_F - P_{G^*}\hat{F} &= \hat{F} - F\hat{H}_F - P_{G^*}(\hat{F} - F\hat{H}_F) - P_{G^*}F\hat{H}_F \\ &= \left[M_{G^*}(\hat{F}\hat{H}_F^{-1} - F) - P_{G^*}F\right]\hat{H}_F,\end{aligned}$$

we get:

$$\begin{aligned}Z'(P_{\tilde{G}^*} - P_{G^*})\hat{F} &= D(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'F\hat{H}_F \\ &\quad + \left\{F'(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)(G^{*\prime}G^*)^{-1}G^{*\prime}\hat{F} + \mathscr{R}_{3,F}(\tilde{G})'\hat{F}\right\} \\ &\quad + D(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'(M_{G^*}(\hat{F}\hat{H}_F^{-1} - F) - P_{G^*}F)\,\hat{H}_F. \end{aligned} \tag{C.29}$$

By plugging equations (C.21), (C.26) and (C.29) into the RHS of equation (C.16), we get:

$$\begin{aligned}\hat{F}\hat{H}_F^{-1} - F &= \eta_F - M_{G^*}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)D' - G^*(G^{*\prime}G^*)^{-1}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'F \\ &\quad - F\left(\frac{\Lambda'\Lambda}{N_H}\right)D\left[\frac{1}{2T}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'F\right]\hat{H}_F\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} \\ &\quad + \mathscr{R}_{1,F}(\hat{F},\tilde{G}) - \mathscr{R}_{2,F}(\tilde{G}) - \mathscr{R}_{3,F}(\tilde{G}) + \mathscr{R}_{4,F}(\hat{F},\tilde{G}),\end{aligned} \tag{C.30}$$

where:

$$\begin{aligned}\mathscr{R}_{4,F}(\hat{F},\tilde{G}) &= -F\left(\frac{\Lambda'\Lambda}{N_H}\right)\left[\frac{1}{2T}F'(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)(G^{*\prime}G^*)^{-1}G^{*\prime}\hat{F} + \frac{1}{2T}\mathscr{R}_{3,F}(\tilde{G})'\hat{F}\right. \\ &\quad \left. + \frac{1}{2T}D(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'(M_{G^*}(\hat{F}\hat{H}_F^{-1} - F) - P_{G^*}F)\,\hat{H}_F\right]\left(\frac{F'\hat{F}}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}.\end{aligned} \tag{C.31}$$

Let us expand the matrix $(F'\hat{F}/2T)^{-1}$ in equation (C.30). By using $\hat{F} = [F + (\hat{F}\hat{H}_F^{-1} - F)]\hat{H}$, we have:

$$\begin{aligned}\left(\frac{F'\hat{F}}{T}\right)^{-1} &= \left[(F'F/T)\left(I_{K_H} + (F'F/T)^{-1}F'(\hat{F}\hat{H}_F^{-1} - F)/T\right)\hat{H}_F\right]^{-1} \\ &= \hat{H}_F^{-1}\left(I_{K_H} + \mathscr{A}(F,\hat{F})\right)^{-1}(F'F/T)^{-1},\end{aligned} \tag{C.32}$$

where $\mathscr{A}(F,\hat{F}) = \left(\frac{F'F}{2T}\right)^{-1}\frac{F'(\hat{F}\hat{H}_F^{-1}-F)}{T}$. Equation (C.32) allows us to rewrite the RHS of equation (C.30) as:

$$\begin{aligned}\hat{F}\hat{H}_F^{-1} - F &= \eta_F - M_{G^*}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)D' - G^*(G^{*\prime}G^*)^{-1}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'F \\ &\quad -F\left(\frac{\Lambda'\Lambda}{N_H}\right)D\left[\frac{1}{2T}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'F\right]\left(\frac{F'F}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} + \mathscr{R}_F(\hat{F},\tilde{G}),\end{aligned} \tag{C.33}$$

where:

$$\mathscr{R}_F(\hat{F},\tilde{G}) = \mathscr{R}_{1,F}(\hat{F},\tilde{G}) - \mathscr{R}_{2,F}(\tilde{G}) - \mathscr{R}_{3,F}(\tilde{G}) + \mathscr{R}_{4,F}(\hat{F},\tilde{G}) - \mathscr{R}_{5,F}(\hat{F},\tilde{G}), \tag{C.34}$$

with:

$$\begin{aligned}\mathscr{R}_{5,F}(\hat{F},\tilde{G}) &= F\left(\frac{\Lambda'\Lambda}{N_H}\right)D\left[\frac{1}{2T}(\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*)'F\right] \\ &\quad \times\left[\left(I_{K_H} + \mathscr{A}(F,\hat{F})\right)^{-1} - I_{K_H}\right]\left(\frac{F'F}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}.\end{aligned} \tag{C.35}$$

(ii) Let us now show the upper bound on $\|\eta_F\|$ given in equation (C.14). From equation (C.13), Lemma S.1 and Assumption H.2, we have:

$$\begin{aligned}\|\eta_F\| &\leq \frac{1}{2N_HT}\left(\|ee'\hat{F}\| + \|F\Lambda'e'\hat{F}\|\right)O_p(1) + O_p\left(\frac{1}{N_H}\|e\Lambda\|\right) \\ &\leq \frac{1}{2N_HT}\left(\|ee'\| + \|F\Lambda'e'\|\right)O_p(\sqrt{T}) + O_p\left(\frac{1}{N_H}\|e\Lambda\|\right),\end{aligned} \tag{C.36}$$

where the last inequality follows from $\hat{F}'\hat{F}/(2T) = I_{K_F}$, as $\hat{F}$ is estimated by PCA. Using inequality (C.36) and Lemma C.3 we get $T^{-1/2}\|\eta_F\| = O_p\left(\frac{1}{\sqrt{\min(N_H,T)}}\right)$.

(iii) Finally, we prove the upper bound on $\|\mathscr{R}_F(\hat{F},\tilde{G})\|$ given in equation (C.15). We bound separately the norm of each term in the RHS of equation (C.34). We use the following result linking the operator norm and the Frobenius norm of a generic $(m \times n)$ matrix A (see, e.g. Horn and Johnson

(2013)):

$$\|A\|_{op} \leq \|A\| \leq \sqrt{\min(m,n)}\ \|A\|_{op}. \tag{C.37}$$

Using $K_H \leq T$ and inequality (C.37), from equation (C.17) we get:

$$\begin{aligned}
\frac{1}{\sqrt{T}}\|\mathscr{R}_{1,F}(\hat{F},\tilde{G})\| &\leq \sqrt{K_H}\frac{1}{\sqrt{T}}\|\mathscr{R}_{1,F}(\hat{F},\tilde{G})\|_{op} \\
&\leq \sqrt{K_H}\left\{2\ \frac{1}{2N_H T}\ \|eX'\|_{op} + \|P_{\tilde{G}^*} - P_{G^*}\|_{op}\ \frac{1}{2N_H T}\ \|XX'\|_{op}\right\} \\
&\quad \times \|P_{\tilde{G}^*} - P_{G^*}\|_{op}\left\|\frac{\hat{F}}{\sqrt{T}}\right\|_{op}\left\|\left(\frac{F'\hat{F}}{2T}\right)^{-1}\right\|_{op}\left\|\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\right\|_{op}.
\end{aligned}$$

Using the result in (C.37) and Lemma S.1 we have:

$$\left\|\left(\frac{F'\hat{F}}{2T}\right)^{-1}\right\|_{op} \leq \left\|\left(\frac{F'\hat{F}}{2T}\right)^{-1}\right\| = O_p(1), \tag{C.38}$$

$$\left\|\left(\frac{B'B}{N_H}\right)^{-1}\right\|_{op} \leq \left\|\left(\frac{B'B}{N_H}\right)^{-1}\right\| = O(1), \tag{C.39}$$

$$\left\|\frac{\hat{F}}{\sqrt{T}}\right\|_{op} \leq \left\|\frac{\hat{F}}{\sqrt{T}}\right\| = O_p(1). \tag{C.40}$$

This allows us to write:

$$\frac{1}{\sqrt{T}}\|\mathscr{R}_{1,F}(\hat{F},\tilde{G})\| = O_p\left(\frac{1}{2N_H T}\|eX'\|_{op}\|P_{\tilde{G}^*} - P_{G^*}\|_{op}\right) + O_p\left(\frac{1}{2N_H T}\|XX'\|_{op}\|P_{\tilde{G}^*} - P_{G^*}\|^2_{op}\right). \tag{C.41}$$

Let us bound each term in the RHS of equation (C.41). Using the expression for $P_{\tilde{G}^*} - P_{G^*}$ in equation (C.24) and the triangular inequality, we have:

$$\begin{aligned}
\|P_{\tilde{G}^*} - P_{G^*}\|_{op} &\leq 2\|M_{G^*}\|_{op}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|_{op}\|(G^{*\prime}G^*)^{-1}G^{*\prime}\|_{op} \\
&\quad + O_p(\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|^2_{op}\|(G^{*\prime}G^*)^{-1}G^{*\prime}\|^2_{op}).
\end{aligned}$$

Moreover we have:

$$\begin{aligned} \|(G^{*\prime}G^*)^{-1}G^{*\prime}\|_{op} &\leq \frac{1}{\sqrt{T}}\left\|\left(\frac{G^{*\prime}G^*}{T}\right)^{-1}\right\|\left\|\frac{G^{*\prime}}{\sqrt{T}}\right\| \\ &= O_p\left(\frac{1}{\sqrt{T}}\right). \end{aligned}$$

This result and $\|M_{G^*}\|_{op} = 1$ allow us to conclude:

$$\begin{aligned} \|P_{\tilde{G}^*} - P_{G^*}\|_{op} &= O_p\left(\frac{1}{\sqrt{T}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|_{op} + \frac{1}{T}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|_{op}^2\right) \\ &= O_p\left(\frac{1}{\sqrt{T}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|\right). \end{aligned} \tag{C.42}$$

Using the definition of X in equation (C.1) and Lemma C.3, we can bound the term $\|eX'\|_{op}$ in the RHS of equation (C.41) as:

$$\begin{aligned} \frac{1}{2N_H T}\|eX'\|_{op} &\leq \frac{1}{2N_H T}\|eX'\| \\ &\leq \frac{1}{2N_H T}\|e\Lambda F'\| + \frac{1}{2N_H T}\|e\Delta G^{*\prime}\| + \frac{1}{2N_H T}\|e\varepsilon'\| \\ &= O_p\bigg(\frac{1}{\sqrt{\min(N_H, T)}}\bigg). \end{aligned} \tag{C.43}$$

From the definition of X in equation (C.1) and Lemma C.2, we can bound the term $\|XX'\|_{op}$ in equation (C.41) as:

$$\begin{aligned} \frac{1}{2N_H T}\|XX'\|_{op} &\leq \frac{1}{2N_H T}\|XX'\| \\ &\leq \frac{1}{N_H T}\|F\Lambda'\Lambda F'\| + \frac{1}{N_H T}\|F\Lambda'\Delta G^{*\prime}\| + \frac{1}{N_H T}\|F\Lambda'\varepsilon'\| \\ &\quad + \frac{1}{N_H T}\|G^*\Delta'\Lambda F'\| + \frac{1}{N_H T}\|G^*\Delta'\Delta G^{*\prime}\| + \frac{1}{N_H T}\|G^*\Delta'\varepsilon'\| \\ &\quad + \frac{1}{N_H T}\|\varepsilon\Lambda F'\| + \frac{1}{N_H T}\|\varepsilon\Delta G^{*\prime}\| + \frac{1}{N_H T}\|\varepsilon\varepsilon'\| \\ &= \frac{1}{N_H T}\|F\Lambda'\Lambda F'\| + \frac{1}{N_H T}\|G^*\Delta'\Delta G^{*\prime}\| + \frac{1}{N_H T}\|\varepsilon\varepsilon'\| \\ &\quad + \frac{1}{N_H T}\|F\Lambda'\Delta G^{*\prime}\| + O_p\bigg(\frac{1}{\sqrt{\min(N_H, T)}}\bigg). \end{aligned} \tag{C.44}$$

The first term in the RHS of the last equation can be bounded as:

$$
\begin{aligned}
\frac{1}{N_H T}\|F\Lambda'\Lambda F'\| &\leq \left\|\frac{F}{\sqrt{T}}\right\|\left\|\frac{\Lambda'\Lambda}{N_H}\right\|\left\|\frac{F}{\sqrt{T}}\right\| \\
&= O_p(1). \qquad \text{(C.45)}
\end{aligned}
$$

The second term in the RHS of equation (C.44) can be bounded as:

$$
\begin{aligned}
\frac{1}{N_H T}\|G^*\Delta'\Delta G^{*\prime}\| &\leq \left\|\frac{G^*}{\sqrt{T}}\right\|\left\|\frac{\Delta'\Delta}{N_H}\right\|\left\|\frac{G^{*\prime}}{\sqrt{T}}\right\| \\
&= O_p(1). \qquad \text{(C.46)}
\end{aligned}
$$

Analogous arguments allow us to bound the remaining terms in the RHS of equation (C.44), and to conclude that:

$$
\frac{1}{2N_H T}\|XX'\|_{op} = O_p(1). \qquad \text{(C.47)}
$$

Collecting results (C.41), (C.42), (C.43) and (C.47) we get:

$$
\begin{aligned}
\frac{1}{\sqrt{T}}\|\mathscr{R}_{1,F}(\hat{F},\tilde{G})\|_{op} &= O_p\left(\frac{1}{\sqrt{\min(N_H,T)}}\left(\frac{1}{\sqrt{T}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|\right)\right) \\
&\quad + O_p\left(\left(\frac{1}{\sqrt{T}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|\right)^2\right). \qquad \text{(C.48)}
\end{aligned}
$$

Let us now bound the term $\mathscr{R}_{2,F}(\hat{F},\tilde{G})$ in equation (C.22). Using $K_H < T$ and inequalities in (C.37) we get:

$$\begin{aligned}
\frac{1}{\sqrt{T}}\|\mathscr{R}_{2,F}(\hat{F},\tilde{G})\| &\leq \sqrt{K_H}\frac{1}{\sqrt{T}}\|\mathscr{R}_{2,F}(\hat{F},\tilde{G})\|_{op} \\
&\leq \sqrt{K_H}\left\|\frac{F}{\sqrt{T}}\right\|_{op}\left\|\frac{\Lambda'\Lambda}{N_H}\right\|_{op}\left\|\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\left(\frac{\Lambda'\Delta}{N_H}\right)-D\right\|_{op} \\
&\quad \times\left\|\frac{G^{*\prime}}{\sqrt{T}}\right\|_{op}\|P_{\tilde{G}^*}-P_{G^*}\|_{op}\left\|\frac{\hat{F}}{\sqrt{T}}\right\|_{op}\left\|\left(\frac{F'\hat{F}}{2T}\right)^{-1}\right\|_{op}\left\|\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\right\|_{op} \\
&\quad +\sqrt{K_H}\left\|\frac{F}{\sqrt{T}}\right\|_{op}\left\|\frac{\Lambda'\varepsilon'}{N_H\sqrt{T}}\right\|_{op}\|P_{\tilde{G}^*}-P_{G^*}\|_{op}\left\|\frac{\hat{F}}{\sqrt{T}}\right\|_{op} \\
&\quad \times\left\|\left(\frac{F'\hat{F}}{2T}\right)^{-1}\right\|_{op}\left\|\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\right\|_{op} \\
&\quad +\sqrt{K_H}\|P_{\tilde{G}^*}-P_{G^*}\|_{op} \\
&\quad \times\left\{\left\|\frac{G^*}{\sqrt{T}}\right\|_{op}\left\|\left(\frac{\Delta'\Lambda}{N_H}\right)\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}-D'\right\|_{op}+\left\|\frac{\varepsilon\Lambda}{N_H\sqrt{T}}\right\|_{op}\left\|\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\right\|_{op}\right\}. \\
&\leq O_p\left(\left\|\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\left(\frac{\Lambda'\Delta}{N_H}\right)-D\right\|_{op}\|P_{\tilde{G}^*}-P_{G^*}\|_{op}\right) \\
&\quad +O_p\left(\left\|\frac{\Lambda'\varepsilon'}{N_H\sqrt{T}}\right\|_{op}\|P_{\tilde{G}^*}-P_{G^*}\|_{op}\right).
\end{aligned}$$

Using assumptions H.1 and H.2, and Lemmas S.1, S.2 and C.2, and equation (C.42) we conclude that:

$$\frac{1}{\sqrt{T}}\|\mathscr{R}_{2,F}(\hat{F},\tilde{G})\| = \frac{1}{\sqrt{N_H}}O_p\left(\frac{1}{\sqrt{T}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1}-G^*\|\right).$$

Analogous arguments allow to bound the term $\mathscr{R}_{3,F}(\tilde{G})$ from equation (C.27) as:

$$\begin{aligned}
\frac{1}{\sqrt{T}}\|\mathscr{R}_{3,F}(\tilde{G})\| &\leq \sqrt{K_H}\frac{1}{\sqrt{T}}\|\mathscr{R}_{3,F}(\tilde{G})\|_{op} \\
&\leq \sqrt{K_H}\frac{1}{\sqrt{T}}\|M_{G^*}\|_{op}\|\tilde{G}^*\tilde{H}_{G^*}^{-1}-G^*\|_{op}\left\|\left(\frac{G^{*\prime}G^*}{T}\right)^{-1}\right\|_{op}\left\|\frac{G^{*\prime}F}{T}\right\|_{op} \\
&\quad +\sqrt{K_H}\frac{1}{\sqrt{T}}\left\|\frac{G^*}{\sqrt{T}}\right\|_{op}\left\|\left(\frac{G^{*\prime}G^*}{T}\right)^{-1}\right\|_{op}\|\tilde{G}^*\tilde{H}_{G^*}^{-1}-G^*\|_{op}\left\|\frac{P_{G^*}F}{\sqrt{T}}\right\|_{op} \\
&\quad +\sqrt{K_H}\frac{1}{\sqrt{T}}\|\mathscr{R}_P(\tilde{G}^*,G^*)Z\|_{op} \\
&\leq \frac{1}{\sqrt{T}}O_p\left(\frac{1}{\sqrt{T}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1}-G^*\|\right)+O_p\left(\left\|\frac{1}{\sqrt{T}}Z\right\|\|\mathscr{R}_P(\tilde{G}^*,G^*)\|\right),
\end{aligned}$$

since:

$$\left\|\frac{P_{G^*}F}{\sqrt{T}}\right\|_{op} \leq \left\|\frac{P_{G^*}F}{\sqrt{T}}\right\| \leq \left\|\frac{G^*}{\sqrt{T}}\right\|\left\|\left(\frac{G^{*\prime}G^*}{T}\right)^{-1}\right\|\left\|\frac{G^{*\prime}F}{T}\right\| = O_p\left(\frac{1}{\sqrt{T}}\right).$$

From the definition of Z in equation (C.20) and Assumption H.1 it follows that:

$$\left\|\frac{1}{\sqrt{T}}Z\right\| = O_p(1).$$

From the bound of $\mathscr{R}_P(\tilde{G}^*, G^*)$ in equation (C.25), we get:

$$\begin{aligned}
\|\mathscr{R}_P(\tilde{G}^*, G^*)\|_{op} &\leq O_p\left(\frac{1}{T}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|^2 \left\|\left(\frac{G^{*\prime}G^*}{T}\right)^{-1}\right\|^2 \left\|\frac{G^{*\prime}}{\sqrt{T}}\right\|^2\right) \\
&= O_p\left(\frac{1}{T}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|^2\right),
\end{aligned}$$

which allows to conclude that:

$$\frac{1}{\sqrt{T}}\|\mathscr{R}_{3,F}(\tilde{G})\| = O_p\left(\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{T}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|\right)\right) + O_p\left(\frac{1}{T}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|^2\right).$$

The term $\mathscr{R}_{4,F}(\hat{F}, \tilde{G})$ can be bounded as:

$$\begin{aligned}
\frac{1}{\sqrt{T}}\|\mathscr{R}_{4,F}(\hat{F}, \tilde{G})\| &\leq \sqrt{K_H}\frac{1}{\sqrt{T}}\|\mathscr{R}_{4,F}(\hat{F}, \tilde{G})\|_{op} \\
&\leq \left\|\frac{F}{\sqrt{T}}\right\|_{op}\left\|\frac{\Lambda'\Lambda}{N_H}\right\|_{op}\Bigg\{\left\|\frac{F}{\sqrt{T}}\right\|_{op}\frac{1}{2\sqrt{T}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|_{op}\left\|\left(\frac{G^{*\prime}G^*}{T}\right)^{-1}\right\|\left\|\frac{G^{*\prime}\hat{F}}{T}\right\|_{op} \\
&\quad + \frac{1}{2\sqrt{T}}\|\mathscr{R}_{3,F}(\tilde{G})\|_{op}\left\|\frac{\hat{F}}{\sqrt{T}}\right\|_{op} \\
&\quad + \|D\|_{op}\frac{1}{2T}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|_{op}\|M_{G^*}\|_{op}\|\hat{F}\hat{H}_F^{-1} - F\|_{op}\|\hat{H}_F\|_{op} \\
&\quad + \frac{1}{2\sqrt{T}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|_{op}\left\|\frac{P_{G^*}F}{\sqrt{T}}\right\|_{op}\|\hat{H}_F\|_{op}\Bigg\} \times \left\|\left(\frac{F'\hat{F}}{2T}\right)^{-1}\right\|_{op}\left\|\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\right\|_{op}. \\
&= O_p\left(\frac{1}{\sqrt{T}}\left(\frac{1}{\sqrt{T}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|\right)\right) + O_p\left(\frac{1}{T}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|\|\hat{F}\hat{H}_F^{-1} - F\|\right) \\
&\quad + O_p\left(\frac{1}{T}\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|^2\right).
\end{aligned}$$

Finally, the term $\mathscr{R}_{5,F}(\hat{F},\tilde{G})$ in equation (C.35) can be bounded as:

$$\begin{aligned}\frac{1}{\sqrt{2T}}\|\mathscr{R}_{5,F}(\hat{F},\tilde{G})\| &\leq \left\|\frac{F}{\sqrt{2T}}\right\|\left\|\frac{\Lambda'\Lambda}{N_H}\right\|\|D\|\left\|\frac{1}{\sqrt{2T}}(\tilde{G}^*\tilde{H}_{G^*}^{-1}-G^*)\right\|\left\|\frac{F}{\sqrt{2T}}\right\| \\ &\quad \times\|(I_{K_H}+\mathscr{A}(F,\hat{F}))^{-1}-I_{K_H}\|\left\|\left(\frac{F'F}{2T}\right)^{-1}\right\|\left\|\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\right\| \\ &\leq O_p\left(\frac{1}{\sqrt{2T}}\|\tilde{G}^*\tilde{H}_{G^*}^{-1}-G^*\|\right)\|(I_{K_H}+\mathscr{A}(F,\hat{F}))^{-1}-I_{K_H}\|.\end{aligned} \tag{C.49}$$

Let us bound the term $(I_{2K_H}+\mathscr{A}(F,\hat{F}))^{-1}-I_{2K_H}$. Assuming that

$$\|\mathscr{A}(F,\hat{F})\| = \left\|\left(\frac{F'F}{2T}\right)^{-1}\frac{1}{2T}F'(\hat{F}\hat{H}_F^{-1}-F)\right\| \leq \rho,$$

for any constant $\rho < 1$, the series representation of the matrix inversion mapping, we have

$$\|(I_{K_H}+\mathscr{A}(F,\hat{F}))^{-1}-I_{K_H}\| \leq \sum_{j=1}^{\infty}\|\mathscr{A}(F,\hat{F})\|^j \leq \frac{1}{1-\rho}\|\mathscr{A}(F,\hat{F})\|.$$

Therefore, we have:

$$\|(I_{K_H}+\mathscr{A}(F,\hat{F}))^{-1}-I_{K_H}\| = O_p(\|\mathscr{A}(F,\hat{F})\|) = O_p\left(\frac{1}{\sqrt{2T}}\|\hat{F}\hat{H}_F^{-1}-F\|\right),$$

which, together with equation (C.49) property (C.37) of the operator norm implies:

$$\frac{1}{\sqrt{2T}}\|\mathscr{R}_{5,F}(\hat{F},\tilde{G})\| = O_p\left(\frac{1}{2T}\|\tilde{G}^*\tilde{H}_{G^*}^{-1}-G^*\|\|\hat{F}\hat{H}_F^{-1}-F\|\right).$$

Q.E.D.

C.1.4 The linearized equation in step 2

Lemma C.5. *We have:*

$$\begin{aligned}\hat{G}\hat{H}_G^{-1} - G &= \eta_G^* - M_{F^*}(\hat{F}^*\hat{H}_{F^*}^{-1} - F^*)W' - F^*(F^{*\prime}F^*)^{-1}(\hat{F}^*\hat{H}_{F^*}^{-1} - F^*)'G \\ &\quad -G\left(\frac{B'B}{N_L}\right)W\left[\frac{1}{T}(\hat{F}^*\hat{H}_{F^*}^{-1} - F^*)'G\right]\left(\frac{G'G}{T}\right)^{-1}\left(\frac{B'B}{N_L}\right)^{-1} \\ &\quad +\mathscr{R}_G^*(\hat{F},\hat{G}),\end{aligned} \tag{C.50}$$

where

$$W = \lim_{N_L\to\infty}\left(\frac{B'B}{N_L}\right)^{-1}\left(\frac{B'\Omega}{N_L}\right) = [W_1 \;\; W_2],$$

the term

$$\eta_G^* = \frac{1}{N_L T}\left(vv'\hat{G} + GB'v'\hat{G}\right)\left(\frac{G'\hat{G}}{T}\right)^{-1}\left(\frac{B'B}{N_L}\right)^{-1} + \frac{1}{N_L}vB\left(\frac{B'B}{N_L}\right)^{-1} \tag{C.51}$$

is such that

$$\|\eta_G^*\|/\sqrt{T} = O_p\left(\frac{1}{\sqrt{\min(N_L,T)}}\right), \tag{C.52}$$

and the reminder term $\mathscr{R}_G^*(\hat{F},\hat{G})$ is such that

$$\begin{aligned}\|\mathscr{R}_G^*(\hat{F},\hat{G})\|/\sqrt{T} &= O_p\left(\frac{1}{\sqrt{\min(N_L,T)}}\|\hat{F}^*\hat{H}_{F^*}^{-1} - F^*\|/\sqrt{T} + (\|\hat{F}^*\hat{H}_{F^*}^{-1} - F^*\|/\sqrt{T})^2\right) \\ &\quad +O_p\left(\frac{1}{T}\|\hat{F}^*\hat{H}_{F^*}^{-1} - F^*\|\|\hat{G}\hat{H}_G^{-1} - G\|\right)\end{aligned} \tag{C.53}$$

Proof: The proof is analogous to the proof of Lemma C.1, and is detailed in the supplementary material.

C.1.5 Writing the linearized equations by components

The recursive equation (C.12) involves the "compound" form $\tilde{G}^*$ of the estimated LF factor in the RHS. Similarly, the recursive equation (C.50) involves the form $\hat{F}^*$ of the estimated HF factor in the

RHS. Let us now rewrite those equations such that their RHS involve estimates $\tilde{G}$ and $\hat{F}$, respectively. This simplifies the combined use of the two equations later on.

By using the definition of F, G^*, M_{G^*} and their estimates, equation (C.12) can be written in components as:

$$
\begin{aligned}
\begin{bmatrix} \hat{F}_1\hat{H}_F^{-1} - F_1 \\ \hat{F}_2\hat{H}_F^{-1} - F_2 \end{bmatrix}
&= \begin{bmatrix} \eta_{F,1} \\ \eta_{F,2} \end{bmatrix} - \begin{bmatrix} M_G & 0 \\ 0 & M_G \end{bmatrix} \begin{bmatrix} \tilde{G}\tilde{H}_G^{-1} - G & 0 \\ 0 & \tilde{G}\tilde{H}_G^{-1} - G \end{bmatrix} \begin{bmatrix} D_1' \\ D_2' \end{bmatrix} \\
&\quad - \begin{bmatrix} G(G'G)^{-1} & 0 \\ 0 & G(G'G)^{-1} \end{bmatrix} \begin{bmatrix} (\tilde{G}\tilde{H}_G^{-1} - G)'F_1 \\ (\tilde{G}\tilde{H}_G^{-1} - G)'F_2 \end{bmatrix} \\
&\quad - \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \left(\frac{\Lambda'\Lambda}{N_H}\right)[D_1 \;\; D_2]\frac{1}{2T} \begin{bmatrix} (\tilde{G}\tilde{H}_G^{-1} - G)'F_1 \\ (\tilde{G}\tilde{H}_G^{-1} - G)'F_2 \end{bmatrix} \left(\frac{F'F}{2T}\right)^{-1} \left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} \\
&\quad + \begin{bmatrix} \mathscr{R}_{F,1}(\hat{F},\tilde{G}) \\ \mathscr{R}_{F,2}(\hat{F},\tilde{G}) \end{bmatrix} \\
&= \begin{bmatrix} \eta_{F,1} \\ \eta_{F,2} \end{bmatrix} - \begin{bmatrix} M_G(\tilde{G}\tilde{H}_G^{-1} - G)D_1' \\ M_G(\tilde{G}\tilde{H}_G^{-1} - G)D_2' \end{bmatrix} - \begin{bmatrix} G(G'G)^{-1}(\tilde{G}\tilde{H}_G^{-1} - G)'F_1 \\ G(G'G)^{-1}(\tilde{G}\tilde{H}_G^{-1} - G)'F_2 \end{bmatrix} \\
&\quad - \frac{1}{2T} \begin{bmatrix} F_1(\Lambda'\Lambda/N_H)D_1 & F_1(\Lambda'\Lambda/N_H)D_2 \\ F_2(\Lambda'\Lambda/N_H)D_1 & F_2(\Lambda'\Lambda/N_H)D_2 \end{bmatrix} \\
&\quad \times \begin{bmatrix} (\tilde{G}\tilde{H}_G^{-1} - G)'F_1 \\ (\tilde{G}\tilde{H}_G^{-1} - G)'F_2 \end{bmatrix} \left(\frac{F'F}{2T}\right)^{-1} \left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} + \begin{bmatrix} \mathscr{R}_{F,1}(\hat{F},\tilde{G}) \\ \mathscr{R}_{F,2}(\hat{F},\tilde{G}) \end{bmatrix}.
\end{aligned}
$$

Therefore we have:

$$\begin{aligned}\hat{F}_1\hat{H}_F^{-1} - F_1 &= \eta_{F,1} - M_G(\tilde{G}\tilde{H}_G^{-1} - G)D_1' - G(G'G)^{-1}(\tilde{G}\tilde{H}_G^{-1} - G)'F_1 \\ &\quad -\frac{1}{2T}F_1\left(\frac{\Lambda'\Lambda}{N_H}\right)\left[D_1(\tilde{G}\tilde{H}_G^{-1} - G)'F_1 + D_2(\tilde{G}\tilde{H}_G^{-1} - G)'F_2\right]\left(\frac{F'F}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} \\ &\quad +\mathscr{R}_{F,1}(\hat{F},\tilde{G}),\end{aligned} \tag{C.54}$$

and:

$$\begin{aligned}\hat{F}_2\hat{H}_F^{-1} - F_2 &= \eta_{F,2} - M_G(\tilde{G}\tilde{H}_G^{-1} - G)D_2' - G(G'G)^{-1}(\tilde{G}\tilde{H}_G^{-1} - G)'F_2 \\ &\quad -\frac{1}{2T}F_2\left(\frac{\Lambda'\Lambda}{N_H}\right)\left[D_1(\tilde{G}\tilde{H}_G^{-1} - G)'F_1 + D_2(\tilde{G}\tilde{H}_G^{-1} - G)'F_2\right]\left(\frac{F'F}{2T}\right)^{-1}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1} \\ &\quad +\mathscr{R}_{F,2}(\hat{F},\tilde{G}).\end{aligned} \tag{C.55}$$

Similarly, by using the definition of F^*, M_{F^*} and their estimates, equation (C.50) can be written as:

$$\begin{aligned}\hat{G}\hat{H}_G^{-1} - G &= \eta_G^* - M_{F^*}[(\hat{F}_1\hat{H}_F^{-1} - F_1)W_1' + (\hat{F}_2\hat{H}_F^{-1} - F_2)W_2'] \\ &\quad -F^*(F^{*\prime}F^*)^{-1}\begin{bmatrix}(\hat{F}_1\hat{H}_F^{-1} - F_1)'G \\ (\hat{F}_2\hat{H}_F^{-1} - F_2)'G\end{bmatrix} \\ &\quad -\frac{1}{T}G\left(\frac{B'B}{N_L}\right)[W_1(\hat{F}_1\hat{H}_F^{-1} - F_1)'G + W_2(\hat{F}_2\hat{H}_F^{-1} - F_2)'G]\left(\frac{G'G}{T}\right)^{-1}\left(\frac{B'B}{N_L}\right)^{-1} \\ &\quad +\mathscr{R}_G^*(\hat{F},\tilde{G}).\end{aligned} \tag{C.56}$$

C.1.6 The system of linearized equations when $K_H = K_L = 1$

Let us now focus on the case with one-dimensional HF and LF factors, i.e., $K_H = K_L = 1$. Then, $\frac{F'F}{2T}$, $\frac{G'G}{T}$, $\hat{H}_F$, $\hat{H}_G$, $\frac{\Lambda'\Lambda}{N_H}$ and $\frac{B'B}{N_L}$ are scalars. Moreover, matrices D and W become (1×2)

matrices:

$$\underset{(1\times 2)}{D} = [d_1 \;\; d_2], \qquad d_j = \lim_{N_H\to\infty}\left(\frac{\Lambda'\Lambda}{N_H}\right)^{-1}\left(\frac{\Lambda'\Delta_j}{N_H}\right), \qquad j = 1,2,$$

$$\underset{(1\times 2)}{W} = [w_1 \;\; w_2], \qquad w_j = \lim_{N_L\to\infty}\left(\frac{B'B}{N_L}\right)^{-1}\left(\frac{B'\Omega_j}{N_L}\right), \qquad j = 1,2.$$

We rename h_H and h_G the scalars H_F and H_G. This allows to re-write the equation for $\hat{F}_1\hat{H}_F^{-1} - F_1$ in (C.54) as:

$$\begin{aligned}
\hat{F}_1\hat{h}_F^{-1} - F_1 &= \eta_{F,1} - d_1 M_G(\tilde{G}\tilde{h}_G^{-1} - G) - G(G'G)^{-1}(\tilde{G}\tilde{h}_G^{-1} - G)'F_1 \\
&\quad -\frac{1}{2T}d_1 F_1(\tilde{G}\tilde{h}_G^{-1} - G)'F_1\left(\frac{F'F}{2T}\right)^{-1} - \frac{1}{2T}d_2 F_1(\tilde{G}\tilde{h}_G^{-1} - G)'F_2\left(\frac{F'F}{2T}\right)^{-1} \\
&\quad +\mathscr{R}_{F,1}(\hat{F},\tilde{G}), \\
&= \eta_{F,1} - \left[d_1 M_G + G(G'G)^{-1}F_1' + \left(\frac{F'F}{2T}\right)^{-1}\frac{d_1}{2T}F_1F_1' + \left(\frac{F'F}{2T}\right)^{-1}\frac{d_2}{2T}F_1F_2'\right] \\
&\quad \times(\tilde{G}\tilde{h}_G^{-1} - G) + \mathscr{R}_{F,1}(\hat{F},\tilde{G}). \qquad \text{(C.57)}
\end{aligned}$$

Similarly, the equation for $\hat{F}_2\hat{H}_F^{-1} - F_2$ in (C.55) becomes:

$$\begin{aligned}
\hat{F}_2\hat{h}_F^{-1} - F_2 &= \eta_{F,2} - \left[d_2 M_G + G(G'G)^{-1}F_2' + \left(\frac{F'F}{2T}\right)^{-1}\frac{d_1}{2T}F_2F_1' + \left(\frac{F'F}{2T}\right)^{-1}\frac{d_2}{2T}F_2F_2'\right] \\
&\quad \times(\tilde{G}\tilde{h}_G^{-1} - G) + \mathscr{R}_{F,2}(\hat{F},\tilde{G}). \qquad \text{(C.58)}
\end{aligned}$$

Let us now consider the equation for the LF factor. From equation (C.56) we have:

$$\begin{aligned}
\hat{G}\hat{h}_G^{-1} - G &= \eta_G^* - M_{F^*}[w_1(\hat{F}_1\hat{h}_F^{-1} - F_1) + w_2(\hat{F}_2\hat{h}_F^{-1} - F_2)] \\
&\quad -F^*(F^{*\prime}F^*)^{-1}\begin{bmatrix} (\hat{F}_1\hat{h}_F^{-1} - F_1)'G \\ (\hat{F}_2\hat{h}_F^{-1} - F_2)'G \end{bmatrix} \\
&\quad -\frac{1}{T}\, G\, [w_1(\hat{F}_1\hat{h}_F^{-1} - F_1)'G + w_2(\hat{F}_2\hat{h}_F^{-1} - F_2)'G]\left(\frac{G'G}{T}\right)^{-1} + \mathscr{R}_G^*(\hat{F}, \hat{G}) \\
&= \eta_G^* - \left[w_1\left(M_{F^*} + \left(\frac{G'G}{T}\right)^{-1}\frac{1}{T}GG'\right) + F^*(F^{*\prime}F^*)^{-1}e_1G'\right](\hat{F}_1\hat{h}_F^{-1} - F_1) \\
&\quad -\left[w_2\left(M_{F^*} + \left(\frac{G'G}{T}\right)^{-1}\frac{1}{T}GG'\right) + F^*(F^{*\prime}F^*)^{-1}e_2G'\right](\hat{F}_2\hat{h}_F^{-1} - F_2) + \mathscr{R}_G^*(\hat{F}, \hat{G}),
\end{aligned} \tag{C.59}$$

where $e_1 = (1, 0)'$ and $e_2 = (0, 1)'$. The last equation can be written as:

$$\hat{G}\hat{h}_G^{-1} - G = \eta_G^* - \mathscr{L}_{G,F_1}(\hat{F}_1\hat{h}_F^{-1} - F_1) - \mathscr{L}_{G,F_2}(\hat{F}_2\hat{h}_F^{-1} - F_2) + \mathscr{R}_G^*(\hat{G}, \hat{F}), \tag{C.60}$$

where:

$$\begin{aligned}
\mathscr{L}_{G,F_1} &= w_1\left(M_{F^*} + \left(\frac{G'G}{T}\right)^{-1}\frac{1}{T}GG'\right) + F^*(F^{*\prime}F^*)^{-1}e_1G', \\
\mathscr{L}_{G,F_2} &= w_2\left(M_{F^*} + \left(\frac{G'G}{T}\right)^{-1}\frac{1}{T}GG'\right) + F^*(F^{*\prime}F^*)^{-1}e_2G',
\end{aligned}$$

and the reminder $\mathscr{R}_G^*(\hat{G}, \hat{F})$ is as in equation (C.53). On the other hand, equations (C.57) and (C.58) can be expressed as:

$$\hat{F}_1\hat{h}_F^{-1} - F_1 = \eta_{F_1} - \mathscr{L}_{F_1,G}\,(\tilde{G}\tilde{h}_G^{-1} - G) + \mathscr{R}_{F,1}(\hat{F}, \tilde{G}), \tag{C.61}$$

$$\hat{F}_2\hat{h}_F^{-1} - F_2 = \eta_{F_2} - \mathscr{L}_{F_2,G}\,(\tilde{G}\tilde{h}_G^{-1} - G) + \mathscr{R}_{F,2}(\hat{F}, \tilde{G}), \tag{C.62}$$

where:

$$
\begin{aligned}
\mathscr{L}_{F_1,G} &= d_1 M_G + G(G'G)^{-1}F_1' + \left(\frac{F'F}{2T}\right)^{-1}\frac{d_1}{2T}F_1F_1' + \left(\frac{F'F}{2T}\right)^{-1}\frac{d_2}{2T}F_1F_2', \\
\mathscr{L}_{F_2,G} &= d_2 M_G + G(G'G)^{-1}F_2' + \left(\frac{F'F}{2T}\right)^{-1}\frac{d_1}{2T}F_2F_1' + \left(\frac{F'F}{2T}\right)^{-1}\frac{d_2}{2T}F_2F_2'.
\end{aligned}
$$

Substituting equations (C.61) and (C.62) in equation (C.60), we get:

$$
\hat{G}h_G^{-1} - G = \eta_G + \mathscr{L}_G(\tilde{G}\tilde{h}_G^{-1} - G) + \mathscr{R}_G(\hat{G}, \tilde{G}, \hat{F}), \tag{C.63}
$$

where:

$$
\eta_G = \eta_G^* - \mathscr{L}_{G,F_1}\eta_{F_1} - \mathscr{L}_{G,F_2}\eta_{F_2}, \tag{C.64}
$$

$$
\mathscr{L}_G = \mathscr{L}_{G,F_1}\mathscr{L}_{F_1,G} + \mathscr{L}_{G,F_2}\mathscr{L}_{F_2,G}, \tag{C.65}
$$

and the reminder term is:

$$
\mathscr{R}_G(\hat{G}, \tilde{G}, \hat{F}) = \mathscr{R}_G^*(\hat{F}, \hat{G}) - \mathscr{L}_{G,F_1}\mathscr{R}_{F,1}(\hat{F}, \tilde{G}) - \mathscr{L}_{G,F_2}\mathscr{R}_{F,2}(\hat{F}, \tilde{G}). \tag{C.66}
$$

We can bound matrix $\mathscr{L}_{G,F_1}$ as:

$$
\begin{aligned}
\|\mathscr{L}_{G,F_1}\|_{op} &\leq |w_1|\Bigg\{\|M_{F^*}\|_{op} + \left\|\left(\frac{G'G}{T}\right)^{-1}\right\|_{op}\left\|\frac{1}{T}GG'\right\|_{op}\Bigg\} \\
&\quad + \left\|\frac{F^*}{\sqrt{T}}\right\|_{op}\left\|\left(\frac{F^{*\prime}F^*}{T}\right)^{-1}\right\|_{op}\|e_1\|_{op}\left\|\frac{G}{\sqrt{T}}\right\|_{op} \\
&= O_p(1).
\end{aligned} \tag{C.67}
$$

Analogous arguments allow to prove that $\|\mathscr{L}_{G,F_2}\|_{op} = O_p(1)$, $\|\mathscr{L}_{F_1,G}\|_{op} = O_p(1)$ and $\|\mathscr{L}_{F_2,G}\|_{op} = O_p(1)$. These results, together with Lemmas C.1 and C.5, allow to bound the term η_G as:

$$
\begin{aligned}
\frac{\|\eta_G\|_{op}}{\sqrt{T}} &\leq \frac{1}{\sqrt{T}}\|\eta_G^*\|_{op} - \|\mathscr{L}_{G,F_1}\|_{op}\frac{1}{\sqrt{T}}\|\eta_{F_1}\|_{op} - \|\mathscr{L}_{G,F_2}\|_{op}\frac{1}{\sqrt{T}}\|\eta_{F_2}\|_{op} \\
&= O_p\left(\frac{1}{\sqrt{\min(N_L, N_H, T)}}\right),
\end{aligned}
$$

and hence:

$$\frac{\|\eta_G\|}{\sqrt{T}} = O_p\bigg(\frac{1}{\sqrt{\min(N_L, N_H, T)}}\bigg). \tag{C.68}$$

Using the results in equations (C.15) and (C.53), the reminder term $\mathscr{R}_G(\hat{G}, \tilde{G}, \hat{F})$ in equation (C.66) can be bounded as:

$$\begin{aligned}
\frac{\|\mathscr{R}_G(\hat{G}, \tilde{G}, \hat{F})\|_{op}}{\sqrt{T}} &\leq \frac{\|\mathscr{R}^*_G(\hat{G}, \hat{F})\|_{op}}{\sqrt{T}} + \|\mathscr{L}_{G,F_1}\|_{op}\frac{\|\mathscr{R}_{F,1}(\hat{F}, \tilde{G})\|_{op}}{\sqrt{T}} + \|\mathscr{L}_{G,F_2}\|_{op}\frac{\|\mathscr{R}_{F,2}(\hat{F}, \tilde{G})\|_{op}}{\sqrt{T}} \\
&= O_p\bigg(\frac{1}{\sqrt{\min(N_L, T)}}\|\hat{F}^*\hat{H}^{-1}_{F^*} - F^*\|/\sqrt{T} + (\|\hat{F}^*\hat{H}^{-1}_{F^*} - F^*\|/\sqrt{T})^2\bigg) \\
&\quad + O_p\bigg(\frac{1}{T}\|\hat{F}^*\hat{H}^{-1}_{F^*} - F^*\|\|\hat{G}\hat{H}^{-1}_G - G\|\bigg) \\
&\quad + O_p\bigg(\frac{1}{\sqrt{\min(N_H, T)}}\|\tilde{G}^*\tilde{H}^{-1}_{G^*} - G^*\|/\sqrt{2T} + (\|\tilde{G}^*\tilde{H}^{-1}_{G^*} - G^*\|/\sqrt{2T})^2\bigg) \\
&\quad + O_p\bigg(\frac{1}{2T}\|\tilde{G}^*\tilde{H}^{-1}_{G^*} - G^*\|\|\hat{F}\hat{H}^{-1}_F - F\|\bigg).
\end{aligned} \tag{C.69}$$

Equations (C.60), (C.61) and (C.62) can be stacked together in the following way:

$$\begin{bmatrix} I_T & 0 & 0 \\ 0 & I_T & 0 \\ \mathscr{L}_{G,F_1} & \mathscr{L}_{G,F_2} & I_T \end{bmatrix}\hat{\upsilon} = \eta_\upsilon + \begin{bmatrix} 0 & 0 & -\mathscr{L}_{F_1,G} \\ 0 & 0 & -\mathscr{L}_{F_2,G} \\ 0 & 0 & 0 \end{bmatrix}\tilde{\upsilon} + \mathscr{R}_\upsilon(\hat{\upsilon}, \tilde{\upsilon}) \tag{C.70}$$

where:

$$\hat{\upsilon} = \begin{bmatrix} \hat{F}_1\hat{h}^{-1}_F - F_1 \\ \hat{F}_2\hat{h}^{-1}_F - F_2 \\ \hat{G}\hat{h}^{-1}_G - G \end{bmatrix}, \qquad \tilde{\upsilon} = \begin{bmatrix} \tilde{F}_1\tilde{h}^{-1}_F - F_1 \\ \tilde{F}_2\tilde{h}^{-1}_F - F_2 \\ \tilde{G}\tilde{h}^{-1}_G - G \end{bmatrix},$$
$$\eta_\upsilon = \begin{bmatrix} \eta_{F_1} \\ \eta_{F_2} \\ \eta^*_G \end{bmatrix}, \qquad \mathscr{R}_\upsilon(\hat{\upsilon}, \tilde{\upsilon}) = \begin{bmatrix} \mathscr{R}_{F,1}(\hat{F}, \tilde{G}) \\ \mathscr{R}_{F,2}(\hat{F}, \tilde{G}) \\ \mathscr{R}^*_G(\hat{G}, \hat{F}) \end{bmatrix}. \tag{C.71}$$

From equations (C.14), (C.15), (C.52), (C.53) we get:

$$\frac{\|\eta_v\|}{\sqrt{T}} = O_p\Big(\frac{1}{\sqrt{\min(N_L, N_H, T)}}\Big), \tag{C.72}$$

$$\frac{\|\mathscr{R}_v(\hat{v}, \tilde{v})\|}{\sqrt{T}} = O_p\Big(\frac{1}{\sqrt{\min(N_L, N_H, T)}} + \Big(\frac{\|\tilde{v}\|}{\sqrt{T}}\Big)^2 + \Big(\frac{\|\hat{v}\|}{\sqrt{T}}\Big)^2\Big). \tag{C.73}$$

Moreover, as

$$\begin{bmatrix} I_T & 0 & 0 \\ 0 & I_T & 0 \\ \mathscr{L}_{G,F_1} & \mathscr{L}_{G,F_2} & I_T \end{bmatrix}^{-1} = \begin{bmatrix} I_T & 0 & 0 \\ 0 & I_T & 0 \\ -\mathscr{L}_{G,F_1} & -\mathscr{L}_{G,F_2} & I_T \end{bmatrix},$$

the system (C.70) and using $\|\tilde{G}^*\tilde{H}_{G^*}^{-1} - G^*\|/\sqrt{T} \leq C$ and $\|\hat{F}^*\tilde{H}_{F^*}^{-1} - F^*\|/\sqrt{T} \leq C$ w.p.a. 1, for some C, can be rewritten as:

$$\hat{v} = \eta_v^\star + \mathscr{L}_v\tilde{v} + \mathscr{R}_v^\star(\hat{v}, \tilde{v}) \tag{C.74}$$

where:

$$\eta_v^\star = \begin{bmatrix} I_T & 0 & 0 \\ 0 & I_T & 0 \\ -\mathscr{L}_{G,F_1} & -\mathscr{L}_{G,F_2} & I_T \end{bmatrix} \eta_v, \qquad \mathscr{R}_v^\star = \begin{bmatrix} I_T & 0 & 0 \\ 0 & I_T & 0 \\ -\mathscr{L}_{G,F_1} & -\mathscr{L}_{G,F_2} & I_T \end{bmatrix} \mathscr{R}_v, \tag{C.75}$$

and

$$\begin{aligned} \mathscr{L}_v &= \begin{bmatrix} I_T & 0 & 0 \\ 0 & I_T & 0 \\ -\mathscr{L}_{G,F_1} & -\mathscr{L}_{G,F_2} & I_T \end{bmatrix} \begin{bmatrix} 0 & 0 & -\mathscr{L}_{F_1,G} \\ 0 & 0 & -\mathscr{L}_{F_2,G} \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -\mathscr{L}_{F_1,G} \\ 0 & 0 & -\mathscr{L}_{F_2,G} \\ 0 & 0 & \mathscr{L}_G \end{bmatrix}, \end{aligned} \tag{C.76}$$

with $\mathscr{L}_G$ defined in equation (C.65). Using result (C.72), we can bound $\tilde{\eta}_v$ as:

$$\frac{\|\eta_v^\star\|_{op}}{\sqrt{T}} \leq \left\| \begin{matrix} I_T & 0 & 0 \\ 0 & I_T & 0 \\ -\mathscr{L}_{G,F_1} & -\mathscr{L}_{G,F_2} & I_T \end{matrix} \right\|_{op} \frac{\|\eta_v\|_{op}}{\sqrt{T}} = O_p\left(\frac{1}{\sqrt{\min(N_L, N_H, T)}}\right), \quad \text{(C.77)}$$

which implies:

$$\frac{\|\eta_v^\star\|}{\sqrt{T}} = O_p\left(\frac{1}{\sqrt{\min(N_L, N_H, T)}}\right). \quad \text{(C.78)}$$

Using analogous arguments, we can bound $\tilde{\mathscr{R}}_v(\hat{v}, \tilde{v})$ as:

$$\frac{\|\mathscr{R}_v^\star(\hat{v}, \tilde{v})\|}{\sqrt{T}} = O_p\left(\frac{1}{\sqrt{\min(N_L, N_H, T)}} + \left(\frac{\|\tilde{v}\|}{\sqrt{T}}\right)^2 + \left(\frac{\|\hat{v}\|}{\sqrt{T}}\right)^2\right). \quad \text{(C.79)}$$

[...]

Let us now compute matrix $\mathscr{L}_G$. We have:

$$\begin{aligned}
\mathscr{L}_G &= \left[w_1\left(M_{F^*} + \left(\frac{G'G}{T}\right)^{-1}\frac{1}{T}GG'\right) + F^*(F^{*\prime}F^*)^{-1}e_1G'\right] \\
&\quad \times\left[d_1M_G + G(G'G)^{-1}F_1' + \left(\frac{F'F}{2T}\right)^{-1}\frac{d_1}{2T}F_1F_1' + \left(\frac{G'G}{T}\right)^{-1}\frac{d_2}{2T}F_1F_2'\right] \\
&\quad +\left[w_2\left(M_{F^*} + \left(\frac{G'G}{T}\right)^{-1}\frac{1}{T}GG'\right) + F^*(F^{*\prime}F^*)^{-1}e_2G'\right] \\
&\quad \times\left[d_2M_G + G(G'G)^{-1}F_2' + \left(\frac{F'F}{2T}\right)^{-1}\frac{d_1}{2T}F_2F_1' + \left(\frac{F'F}{2T}\right)^{-1}\frac{d_2}{2T}F_2F_2'\right] \\
&= w_1d_1M_{F^*}M_G + 2w_1G(G'G)^{-1}F_1' + F^*(F^{*\prime}F^*)^{-1}e_1F_1' \\
&\quad + w_2d_2M_{F^*}M_G + 2w_2G(G'G)^{-1}F_2' + F^*(F^{*\prime}F^*)^{-1}e_2F_2' \\
&\quad + \mathscr{R}_{\mathscr{L}},
\end{aligned}$$

where the reminder term:

$$
\begin{aligned}
\mathscr{R}_{\mathscr{L}} &= -w_1 P_{F^*} G(G'G)^{-1} F_1' \\
&+ w_1 \left(\frac{G'G}{T}\right)^{-1} \left(\frac{F'F}{2T}\right)^{-1} \frac{1}{T} G G' \left[\frac{1}{2T} F_1 F_1' d_1 + \frac{1}{2T} F_1 F_2' d_2\right] \\
&+ F^* \left(\frac{F^{*\prime} F^*}{T}\right)^{-1} \left(\frac{F'F}{2T}\right)^{-1} \left(\frac{G'F_1}{2T}\right) \left[d_1 \left(\frac{e_1 F_1'}{2T}\right) + d_2 \left(\frac{e_1 F_2'}{2T}\right)\right] \\
&- w_2 P_{F^*} G(G'G)^{-1} F_2' \\
&+ w_2 \left(\frac{G'G}{T}\right)^{-1} \left(\frac{F'F}{2T}\right)^{-1} \frac{1}{T} G G' \left[\frac{1}{2T} F_2 F_1' d_1 + \frac{1}{2T} F_2 F_2' d_2\right] \\
&+ F^* \left(\frac{F^{*\prime} F^*}{T}\right)^{-1} \left(\frac{F'F}{2T}\right)^{-1} \left(\frac{G'F_2}{2T}\right) \left[d_1 \left(\frac{e_2 F_1'}{2T}\right) + d_2 \left(\frac{e_2 F_2'}{2T}\right)\right],
\end{aligned}
$$

where we use $M_{F^*} F_j = 0$ for $j = 1, 2$ and $M_G G = 0$. The term $\mathscr{R}_{\mathscr{L}}(F, G)$ can be bounded as:

$$
\|\mathscr{R}_{\mathscr{L}}\| = O_p(T^{-1/2}),
$$

since $\|F_j' G/T\| = O_p(T^{-1/2})$ for $j = 1, 2$. This allows to write:

$$
\begin{aligned}
\mathscr{L}_G &= (w_1 d_1 + w_2 d_2) M_{F^*} M_G \\
&\quad + 2 w_1 G(G'G)^{-1} F_1' + 2 w_2 G(G'G)^{-1} F_2' + P_{F^*} + O_p(T^{-1/2}),
\end{aligned} \tag{C.80}
$$

where $O_p(T^{-1/2})$ denotes a $(T \times T)$ matrix whose norm is $O_p(T^{-1/2})$.

[...]

Since $\|\tilde{v}\|/\sqrt{T} \leq c$, w.p.a. 1, for some constant $c > 0$, the $O_p(T^{-1/2})$ term in the RHS of equation (C.80) can be absorbed into the residual term of equation (C.63). Moreover, by replacing F_1 and F_2 with their residuals in the projection onto G, we modify matrix $\mathscr{L}_G$ by a term of order $O_p(T^{-1/2})$. Hence, we can analyze matrix $\mathscr{L}_G$ as if (F_1, F_2) and G were orthogonal.

[...]

C.1.7 Eigenvalues, eigenvectors and Jordan decomposition of matrix $\mathscr{L}_G$

i) Spectral decomposition of matrix $\mathscr{L}_G$

Let us now compute the eigenvalues and the associated eigenvectors of matrix $\mathscr{L}_G$ defined by:

$$\mathscr{L}_G = P_{F^*} + (w_1 d_1 + w_2 d_2)(M_{F^*} M_G) + 2w_1 G(G'G)^{-1}F_1' + 2w_2 G(G'G)^{-1}F_2'.$$

Since the vectors F_1 and F_2 are orthogonal (asymptotically) to vector G, the matrix $M_{F^*}M_G$ is the orthogonal projection onto the orthogonal complement of the linear subspace generated by vectors F_1, F_2 and G. Moreover the matrix

$$\mathscr{A} = P_{F^*} + 2w_1 G(G'G)^{-1}F_1' + 2w_2 G(G'G)^{-1}F_2'$$

is (asymptotically) idempotent, with $(T-2)$-dimensional null space equal to the orthogonal complement of the span of vectors F_1 and F_2. Hence, matrix $\mathscr{A}$ admits the eigenvalue 1 with multiplicity 2, and the eigenvalue 0 with multiplicity $T-2$. Moreover, matrix $\mathscr{A}$ maps the subspace $\mathscr{E}_1 = span\{F_1, F_2, G\}$ spanned by vectors F_1, F_2 and G into itself. Matrix $\mathscr{A}$ is an oblique projection onto a bi-dimensional subspace of $\mathscr{E}_1$.

We deduce that matrix $\mathscr{L}_G$ admits two invariant subspaces, namely $\mathscr{E}_1$ and its orthogonal complement $\mathscr{E}_2$, of dimensions 3 and $T-3$, respectively. On subspace $\mathscr{E}_2$, the linear operator corresponding to matrix $\mathscr{L}_G$ is diagonal and equal to $w_1 d_1 + w_2 d_2$. On subspace $\mathscr{E}_1$, the linear operators corresponding to matrices $\mathscr{L}_G$ and $\mathscr{A}$ are equal. We conclude that matrix $\mathscr{L}_G$ admits the eigenvalue 0, associated to the eigenvector G, the eigenvalue $w_1 d_1 + w_2 d_2$, with multiplicity $T-3$, associated to the eigenspace $\mathscr{E}_2$, and the eigenvalue 1 with multiplicity 2.

To conclude, let us derive the bi-dimensional eigenspace of matrix $\mathscr{L}_G$ associated to eigenvalue 1. Since this eigenspace is also the eigenspace of matrix A associated to eigenvalue 1, and matrix $\mathscr{A}$ is idempotent, it is enough to find two linearly independent vectors in the image space of $\mathscr{A}$. Two such vectors are:

$$\mathscr{A}F_1 = F_1 + 2(w_1 + w_2\phi)G,$$
$$\mathscr{A}F_2 = F_2 + 2(w_1\phi + w_2)G.$$

ii) Jordan decomposition of matrix $\mathscr{L}_G$

The *Jordan decomposition theorem* [3] ensures the existence of a non-singular $T \times T$ matrix Q and a upper-triangular matrix $\bar{\mathscr{L}}_G$ whose diagonal elements are the eigenvalues of $\mathscr{L}_G$, such that:

$$Q\,\mathscr{L}_G\,Q^{-1} \;=\; \bar{\mathscr{L}}_G, \tag{C.81}$$

where

$$\bar{\mathscr{L}}_G \;=\; \begin{bmatrix} \mathscr{L}_{G,I} & 0 \\ 0 & \mathscr{L}_{G,II} \end{bmatrix} = \left[\begin{array}{cccc|ccc} \lambda^* & 1 & & & & & \\ & \ddots & \ddots & & & \mathbf{0} & \\ & & \lambda^* & 1 & & & \\ & & & \lambda^* & & & \\ \hline & & & & 0 & 0 & 0 \\ & \mathbf{0} & & & & 1 & 1 \\ & & & & & & 1 \end{array}\right], \tag{C.82}$$

where $\lambda^* = w_1 d_1 + w_2 d_2$. The norms of the two matrices $\mathscr{L}_{G,I}$ and $\mathscr{L}_{G,II}$ are $\|\mathscr{L}_{G,I}\|_{op} = w_1 d_1 + w_2 d_2 < 1$ and $\|\mathscr{L}_{G,II}\|_{op} = 1$.

iii) Another decomposition of matrix $\mathscr{L}_G$

We showed that $\mathscr{L}_G$ admits two invariant subspaces, namely $\mathscr{E}_1$ and its orthogonal complement $\mathscr{E}_2$, of dimensions 3 and $T-3$, respectively. Let $[v_1, v_2, v_3]$ orthonormal basis for $\mathscr{E}_1$, and $[w_1, ..., w_{T-3}]$ be an orthonormal basis for $\mathscr{E}_2$. Therefore the matrix defined as:

$$Q = [w_1, ..., w_{T-3}, v_1, v_2, v_3] \tag{C.83}$$

is orthogonal, and unitary as:

$$Q'Q = QQ' = I_T \rightarrow Q = Q^{-1}. \tag{C.84}$$

[3] See theorem 14 in Magnus and Neudecker (2007), p. 18.

C.2 Proof of Proposition 4

[...]

C.3 Proof of Proposition 5

Let $z_t = [f'_{1,t}, f'_{2,t}, g'_t]'$ be the vector of stacked factors at time t, as defined in Section 4.2, and let $\hat{z}_t = [\hat{f}'_{1,t}, \hat{f}'_{2,t}, \hat{g}'_t]'$. From Proposition 4 we have:

$$\frac{1}{T}\sum_{t=1}^{T} \|\hat{z}_t - \hat{H}' z_t\|^2 = O_p\left(\frac{1}{T}\right), \tag{C.85}$$

$$\|\hat{H} - H\| = O_p\left(\frac{1}{\sqrt{T}}\right), \qquad H = I_{2K_H+K_L}. \tag{C.86}$$

C.3.1 Consistency

We recall that the reduced-form factor dynamics is:

$$z_t = C(\theta) z_{t-1} + \zeta_t,$$

where matrix $C(\theta)$ is the autoregressive matrix in Equation (13) written as a function of θ, and $V(\zeta_t) = \Sigma_\zeta(\theta)$. The parameter θ is subject to the constraint $\theta \in \Theta$, where $\Theta \subset \mathbb{R}^p$, is the compact set of parameters values that satisfy matrix equation (5). Parameter θ is estimated by constrained Gaussian Pseudo Maximum Likelihood (PML), and is the solution of the following minimization problem:

$$\hat{\theta} = \arg\max_{\theta\in\Theta} \hat{Q}_T(\theta), \tag{C.87}$$

w.r.t. $\theta \in \Theta$, where the criterion $\hat{Q}_T(\theta)$ is defined as:

$$\hat{Q}_T(\theta) = -\frac{1}{2}\log|\Sigma_\zeta(\theta)| - \frac{1}{2T}\sum_{t=2}^{T} [\hat{z}_t - C(\theta)\hat{z}_{t-1}]' \, \Sigma_\zeta(\theta)^{-1} [\hat{z}_t - C(\theta)\hat{z}_{t-1}]. \tag{C.88}$$

Note that, if the factor values were observable, parameter θ would be estimated by constrained Gaussian PML by minimizing the following criterion:

$$Q_T(\theta) = -\frac{1}{2}\log|\Sigma_\zeta(\theta)| - \frac{1}{2T}\sum_{t=2}^{T}[z_t - C(\theta)z_{t-1}]'\,\Sigma_\zeta(\theta)^{-1}\,[z_t - C(\theta)z_{t-1}]\,. \tag{C.89}$$

w.r.t. $\theta \in \Theta$. Let us rewrite the stacked factor estimate $\hat{z}_t$ as:

$$\hat{z}_t = z_t + (\hat{z}_t - \hat{H}'z_t) + (\hat{H} - H)'z_t. \tag{C.90}$$

Substituting equation (C.90) in the criterion (C.88), using the bounds (C.85) and (C.86) and the uniform boundedness of matrices $C(\theta)$ and $\Sigma_\zeta(\theta)^{-1}$, we get the next Lemma, which is proved in the supplementary material.

Lemma C.6.

$$\hat{Q}_T(\theta) \;=\; Q_T(\theta) + o_p(1), \tag{C.91}$$

uniformly w.r.t. $\theta \in \Theta$.

From standard PML theory (see, for instance, Gourieroux and Monfort (1995)) we have:

$$\sup_{\theta\in\Theta}|Q_T(\theta) - Q_\infty(\theta)| = o_p(1), \tag{C.92}$$

where the limit criterion

$$Q_\infty(\theta) \;=\; -\frac{1}{2}\log|\Sigma_\zeta(\theta)| - \frac{1}{2}E_0\Big[[z - C(\theta)z_{t-1}]'\,\Sigma_\zeta(\theta)^{-1}\,[z_t - C(\theta)z_{t-1}]\Big], \tag{C.93}$$

is minimized uniquely at the true value of parameter θ. Finally, equation (C.92) and Lemma C.6 allo us to conclude that:

$$\sup_{\theta\in\Theta}|\hat{Q}_T(\theta) - Q_\infty(\theta)| = o_p(1). \tag{C.94}$$

Then, by standard results on extremum estimators, we conclude that $\hat{\theta} = \theta + o_p(1)$, i.e. $\hat{\theta}$ is a consistent estimator.

C.3.2 Rate of convergence

The first order conditions (F.O.C.) of the maximization problem (C.87) are:

$$\frac{\partial}{\partial\theta}\hat{Q}_T(\hat{\theta}_T) = 0. \tag{C.95}$$

Applying the mean-value theorem to the F.O.C. in the last equation, we have:

$$\sqrt{T}\frac{\partial}{\partial\theta}\hat{Q}_T(\theta_0) + \frac{\partial^2}{\partial\theta\partial\theta'}\hat{Q}_T(\bar{\theta})\,\sqrt{T}(\hat{\theta}_T - \theta_0) = 0, \tag{C.96}$$

where $\bar{\theta}$ is between θ_0 and $\hat{\theta}_T$ componentwise, and θ_0 denotes the true parameter value. By similar arguments as in Lemma C.6 and equation (C.92) we have the following Lemma, which is proved in the supplementary material:

Lemma C.7.

$$\sup_{\theta\in\Theta}\left\|\frac{\partial^2}{\partial\theta\partial\theta'}\hat{Q}_T(\theta) - \frac{\partial^2}{\partial\theta\partial\theta'}Q_T(\theta)\right\| = o_p(1), \tag{C.97}$$

$$\sup_{\theta\in\Theta}\left\|\frac{\partial^2}{\partial\theta\partial\theta'}Q_T(\theta) - \frac{\partial^2}{\partial\theta\partial\theta'}Q_\infty(\theta)\right\| = o_p(1). \tag{C.98}$$

Moreover since $\hat{\theta}_T$ is consistent, Lemma C.7 implies:

$$\frac{\partial^2}{\partial\theta\partial\theta'}\hat{Q}_T(\bar{\theta}) \;=\; \frac{\partial^2}{\partial\theta\partial\theta'}Q_\infty(\theta_0) + o_p(1), \tag{C.99}$$

where $\frac{\partial^2}{\partial\theta\partial\theta'}Q_\infty(\theta_0)$ is nonsingular. Rearranging equation (C.96) we have:

$$\sqrt{T}(\hat{\theta}_T - \theta_0) \;=\; \left(-\frac{\partial^2}{\partial\theta\partial\theta'}Q_\infty(\theta_0) + o_p(1)\right)^{-1}\sqrt{T}\frac{\partial}{\partial\theta}\hat{Q}_T(\theta_0). \tag{C.100}$$

The term $\sqrt{T}\frac{\partial}{\partial\theta}\hat{Q}_T(\theta_0)$ in the RHS of equation (C.100) can be rewritten as:

$$\sqrt{T}\frac{\partial}{\partial\theta}\hat{Q}_T(\theta_0) = \sqrt{T}\frac{\partial}{\partial\theta}Q_T(\theta_0) + \sqrt{T}\left(\frac{\partial}{\partial\theta}\hat{Q}_T(\theta_0) - \frac{\partial}{\partial\theta}Q_T(\theta_0)\right). \tag{C.101}$$

The first term in the RHS of equation (C.101) can be bounded as:

$$\sqrt{T}\frac{\partial}{\partial\theta}Q_T(\theta_0) \quad = \quad O_p(1), \tag{C.102}$$

applying a CLT for serial dependent data. Results (C.85) and (C.86) allow to bound the second term in the RHS of equation (C.101) as in the next Lemma, which is proved in the supplementary material:

Lemma C.8.

$$\sqrt{T}\left\|\frac{\partial}{\partial\theta}\hat{Q}_T(\theta_0) - \frac{\partial}{\partial\theta}Q_T(\theta_0)\right\| \quad = \quad O_p(1). \tag{C.103}$$

The bounds in equations (C.103) and (C.102) allow to conclude that:

$$\sqrt{T}\|\hat{\theta}_T - \theta_0\| = O_p(1).$$

Q.E.D.

APPENDIX D: Factor dynamics with yearly-quarterly mixed frequencies

In this Appendix we consider the setting with yearly (LF) - quarterly (HF) data, one HF factor and one LF factor (i.e., $K_H = K_L = 1$) as in the empirical section. The model of Section 2 is extended to accommodate $m = 4$ HF subperiods. With scalar factors, the model parameters in the factor dynamics are scalar, and denoted by lower-case letters.

D.1 Structural VAR representation

The dynamics of the stacked factor vector $z_t = [f_{1,t}, f_{2,t}, f_{3,t}, f_{4,t}, g_t]'$ is given by the structural VAR(1) model (Ghysels (2012)):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -r_H & 1 & 0 & 0 & 0 \\ 0 & -r_H & 1 & 0 & 0 \\ 0 & 0 & -r_H & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{1,t} \\ f_{2,t} \\ f_{3,t} \\ f_{4,t} \\ g_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & r_H & a_1 \\ 0 & 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 & a_4 \\ m_1 & m_2 & m_3 & m_4 & r_L \end{bmatrix} \begin{bmatrix} f_{1,t-1} \\ f_{2,t-1} \\ f_{3,t-1} \\ f_{4,t-1} \\ g_{t-1} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \\ v_{3,t} \\ v_{4,t} \\ w_t \end{bmatrix}, \tag{D.1}$$

that is

$$\Gamma z_t = R z_{t-1} + \eta_t, \tag{D.2}$$

where $\eta_t = (v_{1,t}, v_{2,t}, v_{3,t}, v_{4,t}, w_t)'$ is a multivariate white noise process with mean 0 and variance-covariance matrix:

$$\Sigma = \begin{bmatrix} \sigma_H^2 & 0 & 0 & 0 & \sigma_{HL,1} \\ 0 & \sigma_H^2 & 0 & 0 & \sigma_{HL,2} \\ 0 & 0 & \sigma_H^2 & 0 & \sigma_{HL,3} \\ 0 & 0 & 0 & \sigma_H^2 & \sigma_{HL,4} \\ \sigma_{HL,1} & \sigma_{HL,2} & \sigma_{HL,3} & \sigma_{HL,4} & \sigma_L^2 \end{bmatrix}. \tag{D.3}$$

D.2 Restrictions implied by the factor normalization

The factor normalization is:

$$V(z_t) = V\begin{pmatrix} f_{1,t} \\ f_{2,t} \\ f_{3,t} \\ f_{4,t} \\ g_t \end{pmatrix} = \begin{bmatrix} 1 & \phi_1 & \phi_2 & \phi_3 & 0 \\ \phi_1 & 1 & \phi_1 & \phi_2 & 0 \\ \phi_2 & \phi_1 & 1 & \phi_1 & 0 \\ \phi_3 & \phi_2 & \phi_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In particular, under stationarity we have:

$$\phi_1 = Cov(f_{1,t}, f_{2,t}) = Cov(f_{2,t}, f_{3,t}) = Cov(f_{3,t}, f_{4,t}),$$

since $f_{1,t}$, $f_{2,t}$, $f_{3,t}$ and $f_{4,t}$ are consecutive realizations of the HF factor process. Similarly:

$$\phi_2 = Cov(f_{1,t}, f_{3,t}) = Cov(f_{2,t}, f_{4,t}).$$

By computing the variance on both sides of equation (D.2) we get:

$$\Gamma V \Gamma' \quad = \quad RVR' + \Sigma. \tag{D.4}$$

By matrix multiplication:

$$\Gamma V \Gamma' = \begin{bmatrix} 1 & \phi_1 - r_H & \phi_2 - r_H\phi_1 & \phi_3 - r_H\phi_2 & 0 \\ & r_H^2 - 2r_H\phi_1 + 1 & r_H^2\phi_1 - r_H(1+\phi_2) + \phi_1 & r_H^2\phi_2 - r_H(\phi_1+\phi_3) + \phi_2 & 0 \\ & & r_H^2 - 2r_H\phi_1 + 1 & r_H^2\phi_1 - r_H(1+\phi_2) + \phi_1 & 0 \\ & & & r_H^2 - 2r_H\phi_1 + 1 & 0 \\ & & & & 1 \end{bmatrix},$$

and:

$$RVR' = \begin{bmatrix} r_H^2 + a_1^2 & a_1 a_2 & a_1 a_3 & a_1 a_4 & A_{15}^* \\ & a_2^2 & a_2 a_3 & a_2 a_4 & a_2 r_L \\ & & a_3^2 & a_3 a_4 & a_3 r_L \\ & & & a_4^2 & a_4 r_L \\ & & & & A_{55}^* \end{bmatrix},$$

where:

$$\begin{aligned} A_{15}^* &= r_H(\phi_3 m_1 + \phi_2 m_2 + \phi_1 m_3 + m_4) + a_1 r_L, \\ A_{55}^* &= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2\phi_1(m_1 m_2 + m_2 m_3 + m_3 m_4) \\ &\quad +2\phi_2(m_1 m_3 + m_2 m_4) + 2\phi_3 m_1 m_4 + r_L^2. \end{aligned}$$

Hence from (D.4) we get the following equations:

n.	Position	Equation
1	(1,1)	$1 = r_H^2 + a_1^2 + \sigma_H^2$
2	(2,2)	$r_H^2 - 2r_H\phi_1 + 1 = a_2^2 + \sigma_H^2$
3	(3,3)	$r_H^2 - 2r_H\phi_1 + 1 = a_3^2 + \sigma_H^2$
4	(4,4)	$r_H^2 - 2r_H\phi_1 + 1 = a_4^2 + \sigma_H^2$
5	(5,5)	$1 = A_{55}^* + \sigma_L^2$
6	(1,2)	$-r_H + \phi_1 = a_1 a_2$
7	(1,3)	$-r_H\phi_1 + \phi_2 = a_1 a_3$
8	(1,4)	$-r_H\phi_2 + \phi_3 = a_1 a_4$
9	(1,5)	$0 = A_{15}^* + \sigma_{HL,1}$
10	(2,3)	$\phi_1(r_H^2 + 1) - \phi_2 r_H - r_H = a_2 a_3$
11	(2,4)	$\phi_2(r_H^2 + 1) - \phi_3 r_H - \phi_1 r_H = a_2 a_4$
12	(2,5)	$0 = a_2 r_L + \sigma_{HL,2}$
13	(3,4)	$\phi_1(r_H^2 + 1) - \phi_2 r_H - r_H = a_3 a_4$
14	(3,5)	$0 = a_3 r_L + \sigma_{HL,3}$
15	(4,5)	$0 = a_4 r_L + \sigma_{HL,4}$

These equations imply:

$$
\begin{aligned}
&(1) \quad \sigma_H^2 = 1 - r_H^2 - a_1^2,\\
&(2) \quad \sigma_H^2 = r_H^2 - 2r_H\phi_1 + 1 - a_2^2,\\
&(3) \quad \sigma_H^2 = r_H^2 - 2r_H\phi_1 + 1 - a_3^2,\\
&(4) \quad \sigma_H^2 = r_H^2 - 2r_H\phi_1 + 1 - a_4^2,\\
&(5) \quad \sigma_L^2 = 1 - A_{55}^*,\\
&(6) \quad \phi_1 = r_H + a_1a_2,\\
&(7) \quad \phi_2 = r_H^2 + r_Ha_1a_2 + a_1a_3,\\
&(8) \quad \phi_3 = r_H^3 + r_H^2a_1a_2 + r_Ha_1a_3 + a_1a_4,\\
&(9) \quad \sigma_{HL,1} = -A_{15}^*,\\
&(10) \quad \phi_1(r_H^2+1) - \phi_2r_H - r_H = a_2a_3,\\
&(11) \quad \phi_2(r_H^2+1) - \phi_3r_H - \phi_1r_H = a_2a_4,\\
&(12) \quad \sigma_{HL,2} = -a_2r_L,\\
&(13) \quad \phi_1(r_H^2+1) - \phi_2r_H - r_H = a_3a_4,\\
&(14) \quad \sigma_{HL,3} = -a_3r_L,\\
&(15) \quad \sigma_{HL,4} = -a_4r_L.
\end{aligned}
$$

Let θ denote the vector containing r_H, r_L, a_i and m_i for all $i = 1, 2, 3, 4$. Equations (6), (7), (8) express ϕ_1, ϕ_2, ϕ_3 in terms of θ:

$$\phi_1 = r_H + a_1a_2, \tag{D.5}$$

$$\phi_2 = r_H^2 + r_Ha_1a_2 + a_1a_3, \tag{D.6}$$

$$\phi_3 = r_H^3 + r_H^2a_1a_2 + r_Ha_1a_3 + a_1a_4. \tag{D.7}$$

Equations (1), (5), (9), (12), (14) and (15) express the elements of the variance-covariance matrix Σ in terms of θ:

$$\sigma_H^2 = 1 - r_H^2 - a_1^2, \tag{D.8}$$

$$\sigma_L^2 = 1 - A_{55}^*, \tag{D.9}$$

$$\sigma_{HL,1} = -A_{15}^*, \tag{D.10}$$

$$\sigma_{HL,2} = -a_2 r_L, \tag{D.11}$$

$$\sigma_{HL,3} = -a_3 r_L, \tag{D.12}$$

$$\sigma_{HL,4} = -a_4 r_L. \tag{D.13}$$

Finally, the remaining equations (2), (3), (4), (10), (11) and (13) provide restrictions on the elements of θ:

$$r_H^2 - 2r_H\phi_1 + 1 - a_2^2 = 1 - r_H^2 - a_1^2, \tag{D.14}$$

$$a_2^2 = a_3^2 = a_4^2, \tag{D.15}$$

$$\phi_1(r_H^2 + 1) - \phi_2 r_H - r_H = a_2 a_3, \tag{D.16}$$

$$\phi_2(r_H^2 + 1) - \phi_3 r_H - \phi_1 r_H = a_2 a_4, \tag{D.17}$$

$$a_2 a_3 = a_3 a_4. \tag{D.18}$$

By using (D.5), (D.6) and (D.7), the equations (D.14) - (D.18) can be written as:

$$a_2^2 = a_3^2 = a_4^2, \tag{D.19}$$

$$a_2 a_3 = a_3 a_4, \tag{D.20}$$

$$a_1^2 - 2r_H a_2 a_1 - a_2^2 = 0, \tag{D.21}$$

$$a_1 a_2 - r_H a_1 a_3 - a_2 a_3 = 0, \tag{D.22}$$

$$a_1 a_3 - r_H a_1 a_4 - a_2 a_4 = 0. \tag{D.23}$$

The system of equations admits three sets of alternative solutions:

- $a_1 = a_2 = a_3 = a_4 = 0,\ r_H \in \mathbb{R}$,
- $r_H = 0,\ \ a_1 = a_2 = a_3 = a_4 \in \mathbb{R}$,
- $r_H = 0,\ \ a_1 = -a_2 = a_3 = -a_4 \in \mathbb{R}$.

We focus on the first set of solutions, and impose $a_1 = a_2 = a_3 = a_4 = 0$.

D.3 Reduced form representation

By inverting the matrix on the LHS of equation (D.1):

$$\Gamma^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ r_H & 1 & 0 & 0 & 0 \\ r_H^2 & r_H & 1 & 0 & 0 \\ r_H^3 & r_H^2 & r_H & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

the reduced form of the structural VAR(1) model in equation (D.1) is given by (see Ghysels (2012)):

$$\begin{bmatrix} f_{1,t} \\ f_{2,t} \\ f_{3,t} \\ f_{4,t} \\ g_t \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & r_H & a_1 \\ 0 & 0 & 0 & r_H^2 & r_H a_1 + a_2 \\ 0 & 0 & 0 & r_H^3 & r_H^2 a_1 + r_H a_2 + a_3 \\ 0 & 0 & 0 & r_H^4 & r_H^3 a_1 + r_H^2 a_2 + r_H a_3 + a_4 \\ m_1 & m_2 & m_3 & m_4 & r_L \end{bmatrix} \begin{bmatrix} f_{1,t-1} \\ f_{2,t-1} \\ f_{3,t-1} \\ f_{4,t-1} \\ g_{t-1} \end{bmatrix} + \zeta_t,$$

where the zero-mean innovation vector $\zeta_t = \Gamma^{-1}\eta_t$ has the variance-covariance matrix

$$V(\zeta_t) = \begin{bmatrix} \sigma_H^2 & r_H\sigma_H^2 & r_H^2\sigma_H^2 & r_H^3\sigma_H^2 & \sigma_{HL,1} \\ & (1+r_H^2)\sigma_H^2 & r_H(1+r_H^2)\sigma_H^2 & r_H^2(1+r_H^2)\sigma_H^2 & r_H\sigma_{HL,1} + \sigma_{HL,2} \\ & & (1+r_H^2+r_H^4)\sigma_H^2 & r_H(1+r_H^2+r_H^4)\sigma_H^2 & r_H^2\sigma_{HL,1} + r_H\sigma_{HL,2} + \sigma_{HL,3} \\ & & & (1+r_H^2+r_H^4+r_H^6)\sigma_H^2 & r_H^3\sigma_{HL,1} + r_H^2\sigma_{HL,2} + r_H\sigma_{HL,3} + \sigma_{HL,4} \\ & & & & \sigma_L^2 \end{bmatrix}.$$

Let us now impose the restrictions from factor normalization derived in Section D.2. Using $a_1 = a_2 = a_3 = a_4 = 0$, from equations (D.8)-(D.13), the parameters of the variance-covariance matrix of the innovations are:

$$\sigma_H^2 = 1 - r_H^2, \tag{D.24}$$

$$\begin{aligned} \sigma_L^2 &= 1 - [m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2r_H(m_1 m_2 + m_2 m_3 + m_3 m_4) \\ &\quad + 2r_H^2(m_1 m_3 + m_2 m_4) + 2r_H^3 m_1 m_4 + r_L^2\ ], \end{aligned} \tag{D.25}$$

$$\sigma_{HL,1} = -r_H(r_H^3 m_1 + r_H^2 m_2 + r_H m_3 + m_4), \tag{D.26}$$

$$\sigma_{HL,2} = \sigma_{HL,3} = \sigma_{HL,4} = 0. \tag{D.27}$$

D.4 Stationarity conditions

The stationarity condition for the VAR(1) model in equation (D.2) is: the eigenvalues of matrix

$$\Gamma^{-1}R = \begin{bmatrix} 0 & 0 & 0 & r_H & a_1 \\ 0 & 0 & 0 & r_H^2 & r_H a_1 + a_2 \\ 0 & 0 & 0 & r_H^3 & r_H^2 a_1 + r_H a_2 + a_3 \\ 0 & 0 & 0 & r_H^4 & r_H^3 a_1 + r_H^2 a_2 + r_H a_3 + a_4 \\ m_1 & m_2 & m_3 & m_4 & r_L \end{bmatrix}$$

are smaller than one in modulus. If either $a_i = 0$ for all i, or $m_i = 0$ for all i, the stationarity condition becomes: $|r_H| < 1$ and $|r_L| < 1$.